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Asymptotic Bethe ansatz S -matrix and Landau–Lifshitz-type effective 2d actions

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Abstract

Motivated by the desire to relate Bethe ansatz equations for anomalous dimensions found on the gauge-theory side of the AdS/CFT correspondence to superstring theory on $AdS_5 \times S^5$ we explore a connection between the asymptotic S -matrix that enters the Bethe ansatz and an effective two-dimensional quantum field theory. The latter generalizes the standard ‘non-relativistic’ Landau–Lifshitz (LL) model describing low-energy modes of ferromagnetic Heisenberg spin chain and should be related to a limit of superstring effective action. We find the exact form of the quartic interaction terms in the generalized LL-type action whose quantum S -matrix matches the low-energy limit of the asymptotic S -matrix of the spin chain of Beisert, Dippel and Staudacher (BDS). This generalizes to all orders in the ‘t Hooft coupling λ an earlier computation of Klose and Zarembo of the S -matrix of the standard LL model. We also consider a generalization to the case when the spin-chain S -matrix contains an extra ‘string’ phase and determine the exact form of the LL 4-vertex corresponding to the low-energy limit of the ansatz of Arutyunov, Frolov and Staudacher (AFS). We explain the relation between the resulting ‘non-relativistic’ non-local action and the second-derivative string sigma model. We comment on modifications introduced by strong-coupling corrections to the AFS phase. We mostly discuss the $SU(2)$ sector but also present generalizations to the $SL(2)$ and $SU(1|1)$ sectors, confirming universality of the dressing phase contribution by matching the low-energy limit of the AFS-type spin-chain S -matrix with tree-level string-theory S -matrix.

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1. Introduction

To demonstrate the AdS/CFT duality one is to establish a direct relation between the spectrum of the $N = 4$ SYM gauge-theory dilatation operator and the spectrum of quantum string energies in $AdS_5 \times S^5$. There are strong indications that both spectra are described by solutions of certain spin-chain-type Bethe ansatz. In the simplest bosonic sector of the gauge theory, the $SU(2)$ sector, the spin chain is a long-range extension of the ferromagnetic $XXX_{1/2}$ model [1–5]. Its Hamiltonian is known explicitly up to three loops; beyond this order the spin chain is defined by the Bethe ansatz [3–6]

$$e^{ip_k L} = \prod_{j \neq k}^M S(p_k, p_j; \lambda), \quad (1.1)$$

$$S(p_k, p_j; \lambda) = S_1(p_k, p_j; \lambda) e^{i\theta(p_k, p_j; \lambda)}, \quad S_1 = \frac{u_k - u_j + i}{u_k - u_j - i}. \quad (1.2)$$

Here p_j ($j = 1, \dots, M$) are momenta of excitations which at one loop reduce to those diagonalizing the $XXX_{1/2}$ monodromy matrix (i.e. magnons) and u_j are their rapidities related to p_j by [3]

$$u_j = u(p_j; \lambda), \quad u(p; \lambda) \equiv \frac{1}{2} \cot \frac{p}{2} \sqrt{1 + \frac{\lambda}{\pi^2} \sin^2 \frac{p}{2}}, \quad (1.3)$$

where λ is the 't Hooft coupling. $S(p_k, p_j; \lambda)$ is a phase shift [4] due to magnon scattering which one may try to interpret as a two-particle scattering matrix of an integrable two-dimensional field-theory [4, 7–9] whose fundamental excitations correspond to the spin-chain magnons. The momenta $p_j = p_j(\lambda, L, M)$ satisfying (1.1) are also subject to the quantization condition $\sum_{k=1}^M p_k = 2\pi m$ encoding the fact of cyclicity of the trace of the corresponding gauge-theory operators. Then, the energy of the spin-chain state or the anomalous dimension of the corresponding operator is given by

$$E = \sum_{j=1}^M \left(\sqrt{1 + \frac{\lambda}{\pi^2} \sin^2 \frac{p_j}{2}} - 1 \right). \quad (1.4)$$

The factor S_1 in (1.2) is the standard Heisenberg model phase shift which enters also the asymptotic (large L) BDS gauge theory Bethe ansatz [3]. An extra phase θ (common to all sectors [10, 11]) is expected to be present in the exact ansatz which, according to the AdS/CFT correspondence, should match the conjectured Bethe ansatz on the string-theory side [6, 12]. The precise structure of this phase (which at weak coupling should start from 3-loop λ^2 terms [6, 13, 14] and at strong coupling should include quantum string $\frac{1}{\sqrt{\lambda}}$ corrections [15–17]) is a key open problem at the moment [10, 18–21]. In addition to showing that S with correct dressing phase θ does come out of the $AdS_5 \times S^5$ string theory of [22], one is also to provide a string-theory derivation of the dispersion relation (1.3), (1.4) containing the ‘discreteness’ factor $\sin^2 \frac{p}{2}$; important steps in the latter direction were recently made in [10, 23, 24].⁵

To understand how the Bethe ansatz (1.1) may be related to string theory one may try to directly associate with it a two-dimensional action describing magnon interactions. At leading

⁵ On the gauge-theory side, the history of derivation of the square root formula $\sqrt{1 + \frac{\lambda}{\pi^2} \sin^2 \frac{p}{2}}$ for ‘magnon’ energy starts (in the small p limit, $\sin \frac{p}{2} \rightarrow \frac{p}{2}$) with [25]. All-order arguments for the validity of the formula with full $\sin^2 \frac{p}{2}$ were given in [26] (see equation (62) there) and in [27] (where $\sin^2 \frac{p}{2}$ appeared at intermediate steps of the derivation of the BMN relation). More recently, (1.4) was derived [28] using a matrix model obtained by s-wave truncation of SYM theory on S^3 . An interesting geometrical picture found in [28] appears to provide a link to a related discussion on the string side in [23].

(1-loop) order in λ the effective 2d action describing the ‘low-energy’ part of the spectrum of the ferromagnetic $XXX_{1/2}$ model is the non-relativistic Landau–Lifshitz (LL) action. It can be found by taking the continuum limit in the coherent state path integral representation for the Heisenberg model [29, 30]. The S -matrix of the LL model on an infinite line does match [8] the leading term in the small-momentum expansion (i.e. $u_j \rightarrow \frac{1}{2} \cot \frac{p_j}{2} \rightarrow \frac{1}{p_j}$) of the S_1 -factor in (1.2). The same LL action appears as a ‘fast-string’ limit of the classical string action on $R \times S^3$ [30–32]. One can also reconstruct higher order in λ terms in a generalized effective LL action by matching the energies of the Bethe ansatz states with their field-theory counterparts [31, 33–37].

Our interest in this generalized LL action is due to the fact that it may serve as a bridge between quantum string theory and the generalized Bethe ansatz (1.1). We expect that the large $L = J$ limit of a quantum effective string action will be related to an effective LL action that reproduces the spin-chain S -matrix⁶.

Here we will not address in detail the relation to quantum string theory, concentrating as a first step on the correspondence between the scattering phase entering the spin-chain Bethe ansatz and the generalized LL model that reproduces it as its S -matrix. We shall demonstrate that a low-momentum form of the BDS S -matrix S_1 in (1.2) is the same as the quantum S -matrix for an LL-type action with a particular quartic interaction term, thus generalizing to all orders in λ the S -matrix relation [8] between the Heisenberg model and the standard LL action. The fact that (a limit of) the BDS S -matrix can be interpreted as a quantum field theory S -matrix is non-trivial, indicating the existence of a two-dimensional field-theory description behind the asymptotic gauge theory spin chain.

We shall also show that including the AFS [6] phase in (1.2) leads in a similar way of matching the S -matrices to a non-relativistic field-theory model with quartic interaction vertex that matches exactly the one extracted (using the approach of [31]) from the classical string action on $R \times S^3$. This may not be too surprising, given that the AFS Bethe ansatz was obtained by discretizing [6] the classical $R \times S^3$ string Bethe equations of [32], but this relation may help to relate the quantum deformation [15, 19, 20] of the AFS phase to world-sheet quantum corrections in a more direct fashion. In the same spirit, we shall discuss the non-relativistic limit of the AFS-type scattering matrix proposed in [5] for the $SL(2)$ sector and find that it coincides with the tree-level scattering matrix of classical string theory on $AdS_3 \times S^1$. This lends strong support to the idea that the dressing phase relating the ‘gauge’ and ‘string’ Bethe ansätze is universal [10] for all sectors of the theory.

This paper is organized as follows. In section 2, we shall first review the structure of the generalized LL action for the $SU(2)$ sector, first in the $SO(3)$ invariant form and then in the complex scalar form found by expanding near the vacuum state. We shall also discuss the definition of the theory on an infinite line (as required for computation of S -matrix), the role of 2d UV cutoff and its relation to the spin chain. In section 3, we shall illustrate how to compute the tree-level and 1-loop corrections to the 2-particle S -matrix for the generalized LL model containing the all-order kinetic term and few higher derivative interaction terms. We shall follow mostly the same methods as used at the leading order in λ in [8].

In section 4, we shall start with the spin-chain scattering phase in (1.2) and find its low-energy limit in which one keeps only the leading in momentum term at each order in expansion in λ . We shall consider separately the BDS and AFS ansätze and in the latter case we emphasize the new features introduced by the presence of non-trivial corrections to the phase in (1.2). Then in section 5 we shall reconstruct the exact (all-order in λ) quartic vertex in

⁶ It is important to emphasize that quantum corrections computed by quantizing the large- J limit of a classical action need not necessarily be the same as the large- J limit of corrections found from the quantum effective action.

the generalized LL action and show that the resulting quantum field-theory S -matrix matches exactly the low-energy limit of the BDS spin-chain scattering phase.

In section 6, we shall comment on a generalization to larger (compact) sectors containing $SU(2)$ sector. In section 7, we shall discuss a relation between a non-relativistic LL-type action reconstructed from the S -matrix of the AFS ansatz and string theory action on $R \times S^3$, on $AdS_3 \times S^1$ and the fermionic action [53] obtained by truncation of the full superstring action [22] to two fermionic fields corresponding to $SU(1|1)$ sector. We shall show that the corresponding tree-level string S -matrices matches the low-energy, strong coupling limit of the AFS-type S -matrix in the $SU(2)$, $SL(2)$ and $SU(1|1)$ sectors, respectively. We shall also explain that a specific non-local structure of the quartic interaction term in the LL action has its origin in the elimination of the negative-energy modes when passing from a second-derivative to a non-relativistic first-derivative action.

Section 8 will contain some concluding remarks. In appendix A, we shall present the results for the quartic interaction vertex in the LL actions corresponding to the $SU(1|1)$ and $SL(2)$ spin-chain sectors described by the BDS-type Bethe ansatz. In appendix B, we shall give some details of the small momentum expansion of the leading quantum correction [15, 19] to the AFS phase.

2. General structure of the effective Landau–Lifshitz-type action

The LL-type action we will be interested in appears in the description the low-energy modes of the ferromagnetic $SU(2)$ gauge theory spin chain. Its derivation from the spin-chain Hamiltonian involves several steps [31, 34]. First, the quantum-mechanical path integral is expressed in terms of spin coherent states parametrized by a unit 3-vector \vec{n}_a at each site $a = 1, \dots, J$.⁷ The resulting (discrete) action contains a WZ-type (Berry phase) term [29] linear in the time derivative of \vec{n}_a and a Hamiltonian part $\sum_{a=1}^J [\lambda(n_{a+1} - n_a)^2 + O(\lambda^2)]$. One then considers the large- J region and takes the continuum limit by truncating away all but the low-energy spin wave excitations of the periodic chain; only the leading lowest-derivative terms are kept at each order in λ . It turns out then that λ combines with powers of J into an effective parameter $\tilde{\lambda} = \frac{\lambda}{J^2}$ which manifests the existence (at least in the first few orders of expansion in λ) of a scaling BMN-type limit. Furthermore, J then appears in front of the action, implying that for fixed $\tilde{\lambda}$ the large- J limit is the same as the semiclassical limit, with $1/J$ corrections playing the role of quantum corrections to the classical LL model.

2.1. $O(3)$ invariant \vec{n} -field action

The resulting action has the following structure ($\partial_0 = \partial_t$, $\partial_1 \equiv \partial_\sigma$, $\vec{n}^2 = 1$),

$$S = J \int dt \int_0^{2\pi} \frac{d\sigma}{2\pi} \mathcal{L}, \quad \mathcal{L} = \vec{C}(n) \cdot \partial_0 \vec{n} - \mathcal{H}(\partial_1 n), \quad (2.1)$$

$$\mathcal{H} = \mathcal{H}_2 + \sum_{k=2}^{\infty} \mathcal{H}_{2k}, \quad \mathcal{H}_{2k} \sim \left(\frac{\lambda}{J^2}\right)^k (\partial_1^{2k} n^4 + \dots + \partial_1^{2k} n^{2k}), \quad (2.2)$$

where J is the total spin equal to the spin-chain length L and $\vec{C}(n)$ is the same as a monopole potential on S^2 , i.e. $dC = \epsilon^{ijk} n_i dn_j \wedge dn_k$. The general form of the ‘kinetic’ part \mathcal{H}_2 can be found [31, 33] from the continuum limit of the coherent state expectation value of the leading

⁷ \vec{n} represents two ‘phase-space’ variables of the ‘classical spin’ $U^* \vec{\sigma} U = \vec{n}$. On the string side \vec{n} corresponds to the two transverse modes of a ‘fast’ string on $R \times S^3$.

spin–spin part of the gauge theory dilatation operator [2, 41] (assuming consistency with the BMN limit which is also implied in (1.1) and (1.2))

$$\mathcal{H}_2 = \frac{1}{4}\tilde{n}\left(\sqrt{1 - \tilde{\lambda}\partial_1^2} - 1\right)\tilde{n}, \quad \tilde{\lambda} \equiv \frac{\lambda}{J^2}. \quad (2.3)$$

The ‘2-loop’ [31], ‘3-loop’ [35, 36] and ‘4-loop’ [37] terms in \mathcal{H} are

$$\mathcal{H}_4 = \frac{1}{32}a_1\tilde{\lambda}^2(\partial_1\tilde{n})^4, \quad (2.4)$$

$$\mathcal{H}_6 = \frac{1}{64}\tilde{\lambda}^3[b_1(\partial_1\tilde{n})^2(\partial_1^2\tilde{n})^2 + b_2(\partial_1\tilde{n}\partial_1^2\tilde{n})^2 + b_3(\partial_1\tilde{n})^6], \quad (2.5)$$

$$\begin{aligned} \mathcal{H}_8 = \tilde{\lambda}^4 & [c_1(\partial_1^2\tilde{n})^4 + c_2(\partial_1\tilde{n})^2(\partial_1^2\tilde{n}\partial_1^4\tilde{n}) + c_3(\partial_1\tilde{n}\partial_1^5\tilde{n})(\partial_1\tilde{n})^2 + c_4(\partial_1^3\tilde{n})^2(\partial_1\tilde{n})^2 \\ & + c_5(\partial_1\tilde{n})^4(\partial_1^2\tilde{n})^2 + c_6(\partial_1\tilde{n}\partial_1^2\tilde{n})^2(\partial_1\tilde{n})^2 + c_7(\partial_1\tilde{n})^8]. \end{aligned} \quad (2.6)$$

The known coefficients consistent with the leading terms in the gauge-theory dilatation operator [2, 41] and thus with the BDS ansatz (i.e. (1.1) with $S = S_1$) are [31, 36, 37]

$$a_1 = \frac{3}{4}, \quad b_1 = -\frac{7}{4}, \quad b_2 = -\frac{23}{2}, \quad b_3 = \frac{3}{4}, \quad (2.7)$$

$$c_5 = \frac{111}{4096}, \quad c_7 = -\frac{267}{32768}, \quad c_1 - c_2 + c_3 + c_4 = -\frac{59}{2048}. \quad (2.8)$$

While the coefficients a_1 and b_1 appear to be non-renormalized when going from small to large λ region, the coefficients b_2, b_3 and at least c_5 and c_7 are, in fact, functions of λ [15, 36]. They have unequal values at $\lambda \rightarrow 0$ and $\lambda \rightarrow \infty$, i.e. they are found to be different from the BDS coefficients in (2.7) when one starts from the ‘string’ AFS Bethe ansatz (which includes a non-trivial phase θ [6] in S in (1.1)). The ‘string’ values that agree with the classical string-theory predictions are [31, 36, 37]

$$b_2 = -\frac{25}{2}, \quad b_3 = \frac{13}{16}, \quad c_5 = \frac{119}{4096}, \quad c_7 = -\frac{323}{32768}. \quad (2.9)$$

2.2. Complex scalar form of the action

One may solve the constraint $\tilde{n}^2 = 1$, i.e. $n_s = \sqrt{1 - n_s n_s}$ ($s = 1, 2$) and express the action (2.1) in terms of the two independent ‘magnon’ fields n_s whose fluctuations describe deviations from the ferromagnetic vacuum $\tilde{n} = (0, 0, 1)$ representing the gauge-theory BPS state $\text{tr}Z^J$. As already mentioned, since J appears in front of the action (2.1) defined on a circle of radius 1, the large- J expansion for fixed $\tilde{\lambda}$ and fixed length of the string represents quantum loop expansion of the LL model. Keeping also the excitation number of a magnon state fixed, these $1/J$ quantum corrections to the energies of the LL states then match finite-size corrections computed directly from the Bethe ansatz [35, 36, 42].

Our aim here will be to compute the magnon S -matrix from the LL model and to compare it to the spin-chain scattering phase S in (1.2). For this purpose, a different limit is appropriate, in which the LL model is defined on an infinite line [8]. This can be accomplished by taking $J \rightarrow \infty$ while keeping the ‘t Hooft coupling λ and the magnon momenta fixed⁸. As follows from the structure of (2.1), (2.2), rescaling the spatial coordinate

$$x = \frac{J}{2\pi}\sigma, \quad \sigma \in (0, 2\pi), \quad (2.10)$$

⁸ Similar limit was considered in [7] and in connection with the antiferromagnetic state of spin chain [38–40]. Recently it was emphasized also in [23, 24].

and making the field redefinition [35]

$$n_s = 2\sqrt{1 - z^2 z_s}, \quad \phi \equiv z_1 + iz_2, \quad (2.11)$$

we can rewrite the LL action (2.1) as a ‘first-order’ action for a complex scalar ‘magnon’ field ϕ

$$\mathcal{S} = \int dt \int_0^J dx \{ \phi^* [i\partial_t - (\sqrt{1 - \tilde{\lambda}\partial_x^2} - 1)]\phi - V(\phi, \phi^*) \}, \quad (2.12)$$

$$\tilde{\lambda} \equiv \frac{\lambda}{(2\pi)^2}. \quad (2.13)$$

Here V contains terms of all orders in powers of ϕ and its spatial derivatives and depends only on λ and not on J :

$$V = V_4 + V_6 + \dots, \quad V_{2n} \sim \sum_{k=1}^{\infty} \tilde{\lambda}^k \partial_x^{2k} (\phi^* \phi)^n. \quad (2.14)$$

The dependence on J is now only in the length of the spatial direction and thus $J \rightarrow \infty$ corresponds to a theory on an infinite line (provided we also scale the quantum numbers m_k of modes on a circle so that momenta $p_k = \frac{2\pi m_k}{J}$ stay fixed in the limit).

Explicitly, the leading quartic interaction term V_4 originating from the first three terms in \mathcal{H} in (2.3) and (2.4) has the form ($\phi' = \partial_x \phi$)

$$V_4 = |\phi|^2 \sqrt{1 - \tilde{\lambda}\partial_x^2} |\phi|^2 - \frac{1}{2} |\phi|^2 (\phi^* \sqrt{1 - \tilde{\lambda}\partial_x^2} \phi + \text{c.c.}) + \frac{1}{2} a_1 \tilde{\lambda}^2 |\phi'|^4 + \frac{1}{16} \tilde{\lambda}^3 [2(2b_1 + b_2) |\phi'|^2 |\phi''|^2 + b_2 (\phi''^2 \phi'^{*2} + \text{c.c.})] + \mathcal{O}(\tilde{\lambda}^4), \quad (2.15)$$

or, expanded in $\tilde{\lambda}$ to ‘4-loop’ order,

$$\begin{aligned} V_4 = & \frac{\tilde{\lambda}}{4} (\phi^{*2} \phi'^2 + \text{c.c.}) \\ & - \frac{\tilde{\lambda}^2}{8} \left[\frac{1}{2} |\phi|^2 (\phi'''' \phi^* + \text{c.c.}) + 4 |\phi|^2 (\phi''' \phi'^* + \text{c.c.}) + 6 |\phi''|^2 |\phi|^2 - 4a_1 |\phi'|^4 \right] \\ & - \frac{\tilde{\lambda}^3}{4} \left[\frac{1}{8} |\phi|^2 (\phi^{(6)} \phi^* + \text{c.c.}) + \frac{3}{2} |\phi|^2 (\phi^{(5)} \phi'^* + \text{c.c.}) + \frac{15}{4} |\phi|^2 (\phi^{(4)} \phi''^* + \text{c.c.}) \right. \\ & \left. + 5 |\phi|^2 |\phi'''|^2 - \frac{1}{2} (2b_1 + b_2) |\phi'|^2 |\phi''|^2 - \frac{1}{4} b_2 (\phi''^2 \phi'^{*2} + \text{c.c.}) \right] + \mathcal{O}(\tilde{\lambda}^4). \end{aligned} \quad (2.16)$$

The action (2.12) has manifest $U(1)$ symmetry and ‘hidden’ $O(3)$ symmetry (which was explicit in (2.1)). Since we expect this action to describe an integrable field theory, the quartic interaction term may effectively determine all higher order terms (modulo field redefinitions): the S -matrix should factorize and thus should be obtainable from bubble graphs with quartic interactions only, just as in the leading-order LL action case discussed in [8].

2.3. Infinite line limit and small momentum expansion

In section 3, we shall first consider the tree-level 2-particle S -matrix for the action (2.12), (2.16) and then also compute the first few terms in its loop expansion. We shall find that the results for the choice of coefficients in (2.7) and (2.8) match the low-momentum limit of the small λ expansion of the BDS S -matrix in (1.2). Then in section 4 we shall consider the opposite problem of reconstructing higher order terms in (2.15) and (2.16) by starting with a low-momentum limit of the full BDS S -matrix.

To prepare for this discussion, it is important to clarify the nature of limits we will be taking and also the role of the 2d field theory cutoff in this context. To consider the S -matrix, we should ignore the periodicity condition in the spatial coordinate and define the field theory on an infinite line. Formally, it may seem that this can be achieved by sending J in (2.12) to infinity but this ignores the presence of a hidden UV scale in the problem. An indication of a need for a spatial scale can be seen, e.g., from the fact that x in (2.12) does not have the standard length dimension ($\tilde{\lambda}$ and J should be dimensionless).

Let us go back to the spin-chain picture and consider the limit in which the number of sites J is sent to infinity while the periodicity condition is not imposed. In that case we get an infinite 1d lattice whose spacing may be denoted as a . For a finite number of points J of a periodic chain of length L the step of the lattice is $a = \frac{L}{J}$. The limit we are interested in is when both L and J are sent to infinity with a kept finite. More precisely, it is the dimensionless product ap where p is a one-dimensional momentum (with canonical mass dimension) that should be kept finite. The momenta of magnons on a circle are $p_k = \frac{2\pi n_k}{L} = \frac{2\pi n_k}{aJ}$ and they remain finite provided n_k is also scaled to infinity together with J .

Next, if we take a continuum limit $a \rightarrow 0$ of the spin-chain Hamiltonian on an infinite lattice (using that $\tilde{n}_{x+a} - \tilde{n}_x = a\tilde{n}' + \frac{1}{2}a^2\tilde{n}'' + \dots$, etc) the result will differ from (2.12) by a rescaling $x \rightarrow a^{-1}x$. Then ∂_x will be replaced by $a\partial_x$ and there will be a factor of $\frac{1}{a}$ in front of the action (coming from the integration measure). Higher derivative terms will be suppressed by higher powers of a ; most of them can be ignored assuming that one keeps only the leading in a term at each order of expansion in λ . Note that the presence of the UV cutoff factor $\frac{1}{a}$ in front of the action is natural on power counting grounds: the standard loop expansion of the leading-order LL action contains linear UV divergences [42, 43]. One may choose to ignore all power divergences using, e.g., the zeta-function or dimensional regularization prescription as in [8].⁹ Then a will play the role of an effective coupling or an effective 2d Planck constant that counts loop order.

If we ignore all power divergences then the field-theory S -matrix will involve only dimensionless products (ap, ap') of the scale a and momenta. In the continuum limit, it is natural to expect that it will match the spin-chain S -matrix only in the region when momenta are small compared to the cutoff. Indeed, in the small momentum expansion both p and p' are small compared to the cutoff scale a^{-1} , i.e. $ap \rightarrow 0, ap' \rightarrow 0$ but their ratio p/p' is fixed. Taking this limit can be formally implemented by scaling a to zero while assuming that $\tilde{\lambda}(ap)^2$ is kept finite. This does not necessarily mean that λ is taken to be large: this means only that one wants to keep the leading in $ap \rightarrow 0$ expansion term at each order in expansion in λ , i.e. the limit of small ap is taken *before* the limit of small λ .

The momenta p_i in the spin-chain expressions (1.2), (1.3) are dimensionless, corresponding to the choice of $a = 1$, i.e. of unit step of the lattice. Then p_i in (1.3) should stand for ap_i if we want p_i to have canonical mass dimension. Taking $a \rightarrow 0$ corresponds to uniformly scaling all momenta to zero, so that

$$u = \frac{1}{2} \cot \frac{ap}{2} \sqrt{1 + \frac{\lambda}{\pi^2} \sin^2 \frac{ap}{2}} \rightarrow \left(\frac{1}{ap} + \dots \right) \sqrt{1 + \frac{\lambda}{\pi^2} [(ap)^2 + \dots]}. \quad (2.17)$$

We shall discuss such an expansion of the spin-chain S -matrix (1.2) in section 4.

Let us mention also the analogy of this limit of the spin-chain S -matrix with the BMN-type scaling limit in the Bethe ansatz equations (1.1). Suppose we take the large length $L = J \gg 1$ limit in (1.1) by rescaling at the same time the momenta so that the lhs of (1.1) stays finite,

⁹ Such a prescription that ignores all power divergences appears to be necessary in order to match the BDS S -matrix (see section 5). It is also consistent with the expected conformal invariance of the dual string theory, predictions of which we should eventually match by starting with a properly modified AFS ansatz.

$p_k = \frac{\tilde{\beta}'_k}{J}$, i.e. $\tilde{\beta}'_k$ will be finite in the limit. Then $u(p)$ in (1.3) that enters the scattering phase (1.2) will become

$$u = J\bar{u} + \dots, \quad \bar{u} = \frac{1}{\tilde{\beta}'} \sqrt{1 + \frac{\lambda}{(2\pi)^2 J^2} \tilde{\beta}'^2} \quad (2.18)$$

and thus

$$S_1(p', p) \rightarrow \hat{S}_1(p', p) = \frac{\bar{u}(p') - \bar{u}(p) + iJ^{-1}}{\bar{u}(p') - \bar{u}(p) - iJ^{-1}}. \quad (2.19)$$

There is then a direct analogy with the discussion above with the role of a played by J^{-1} , assuming that we keep only the leading term in the J^{-1} expansion at each order in the small λ expansion. This is formally the same as keeping $\tilde{\lambda} \equiv \frac{\lambda}{J^2}$ fixed while taking J to be large. Expanding $\hat{S}_1(p', p)$ in powers of J^{-1} will be analogous to the small momentum expansion or quantum loop expansion in the corresponding effective field theory. The Bethe ansatz equations (1.1) make sense of course only for the theory on a circle, implying that at leading order in J^{-1} one has $e^{i\tilde{\beta}'_k} = 1$, i.e. $\tilde{\beta}'_k = 2\pi n_k + O(J^{-1})$.¹⁰

One may wonder if it is possible to extend the matching between the two-dimensional field-theory S -matrix and the spin-chain S -matrix by keeping all the higher derivative terms in the kinetic term ($\vec{n}_{x+a} - \vec{n}_x = 2 \sinh \frac{a\partial_x}{2} \vec{n}_{x+\frac{a}{2}}$) but still replacing the lattice sum by an infinite integral and $\vec{n}_x(t)$ by a continuous field $\vec{n}(t, x)$ (with J assumed to be taken to infinity so that the theory is defined on an infinite line). In this case the kinetic term in (2.3) or in (2.12) will be replaced by its ‘discreet’ counterpart [36]:

$$i\partial_t - \left(\sqrt{1 - 4\tilde{\lambda} \sinh^2 \frac{a\partial_x}{2}} - 1 \right). \quad (2.20)$$

The corresponding dispersion relation is the same as for the spin-chain magnons: $\omega = \sqrt{1 + 4\tilde{\lambda} \sin^2 \frac{ap}{2}} - 1$. Moreover, the range of momenta is restricted to $(-\frac{\pi}{a}, \frac{\pi}{a})$, so that the loop integrals should be automatically finite for a finite cutoff a . One may then try to fix the quartic interaction in the corresponding analogue of (2.12) so that to match the spin-chain S -matrix beyond the small a or low-momentum limit. We will not attempt to do this here. One conceptual issue is that if one does not use the small a expansion, it is not clear how to reinterpret the BDS S -matrix as a sum of bubble graphs in field theory, following the LL example of [8]. One possibility is that the resulting action may be considered as a quantum effective action, whose *tree level* S -matrix should then match the exact spin-chain S -matrix in¹¹.

¹⁰ In [8] the logic was to start with the LL model on a line, derive the corresponding quantum S -matrix $\frac{1/p_k - 1/p_j + i}{1/p_k - 1/p_j - i}$, and then use it in the Bethe ansatz equations like (1.1) with $e^{ip_k J}$ on the lhs. The main observation was that the resulting Bethe ansatz is the same as the limit of the Heisenberg model Bethe ansatz in which the (dimensionless) momenta p_j are taken to be small compared to 1. Indeed, the resulting solutions for low-energy modes found from the two Bethe ansätze are then the same in the large- J limit (up to order $1/J^2$ terms).

¹¹ This interpretation may be useful in order to make contact with string theory: presumably, such action may be derived by taking large- J limit in the quantum string effective action for a string moving on S^3 part of $AdS_5 \times S^5$, just like the classical LL model followed from the classical string action [30, 31]. We shall return to the discussion of related issues in section 7.

3. Field-theory S -matrix

The quadratic part of the action (2.12) resembles the action for the positive-energy part of a massive relativistic scalar field in two dimensions. Indeed, the classical solutions in the free-field limit are¹²

$$\phi(x, t) = \int \frac{dp}{\sqrt{2\pi}} a_p e^{-i\omega_p t + i p x}, \quad \phi^*(x, t) = \int \frac{dp}{\sqrt{2\pi}} a_p^* e^{i\omega_p t - i p x} \quad (3.1)$$

where¹³

$$\omega_p = e(p) - 1, \quad e(p) \equiv \sqrt{1 + \tilde{\lambda} p^2}. \quad (3.2)$$

In the quantum-theory $[a_p, a_{p'}^\dagger] = \delta(p - p')$. The interaction term V in (2.12), however, depends only on spatial derivatives implying that the S -matrix is not expected to be relativistic-invariant. A possible approach to finding a similar action from string theory is to solve for half of modes at the classical level [30, 31] or effectively to integrate them out at the quantum level (see section 7).

To compute the S -matrix corresponding to (2.12) we follow the same steps as in case of the leading-order LL action in [8]. The crucial simplifying point is that the propagator can be chosen as the retarded one, $D(x, t) \sim \theta(t)$, i.e.

$$D(\omega, p) = \frac{i}{\omega - \omega_p + i\epsilon}. \quad (3.3)$$

This implies that the 2-body S -matrix is a sum of bubble diagrams with V_4 in (2.14) as vertices. Let us consider the 2-body scattering process with the initial state $|pp'\rangle = a_p^\dagger a_{p'}^\dagger |0\rangle$ (initial particles being 2 magnons with momenta p, p'), and the final state as $|kk'\rangle = a_k^\dagger a_{k'}^\dagger |0\rangle$. The 2-body scattering matrix is

$$\langle kk' | \hat{S} | pp' \rangle = \langle kk' | T e^{-i \int dt dx V_4} | pp' \rangle. \quad (3.4)$$

As usual, the translational invariance of the action implies momentum conservation, i.e. that $\langle kk' | \hat{S} | pp' \rangle$ is proportional to $\delta^{(2)}(k^\mu + k'^\mu - p^\mu - p'^\mu)$. In two dimensions, the energy and momentum conservation allow the two particles to only exchange their momenta, so that the energy–momentum conservation delta-function becomes

$$\delta(\omega_p + \omega_{p'} - \omega_k - \omega_{k'}) \delta(p + p' - k - k') = K(p, p') \delta_+(p, p', k, k'), \quad (3.5)$$

$$\delta_+ \equiv \delta(p - k) \delta(p' - k') + \delta(p - k') \delta(p' - k), \quad K(p, p') = \frac{1}{\frac{d\omega_p}{dp} - \frac{d\omega_{p'}}{dp'}}. \quad (3.6)$$

Using (3.2), we get

$$K(p, p') = \frac{\tilde{\lambda}^{-1} e(p) e(p')}{p e(p') - p' e(p)} = \frac{\tilde{\lambda}^{-1}}{p - p'} \left[1 + \frac{1}{2} \tilde{\lambda} (p^2 + p'^2 + pp') + \frac{1}{8} \tilde{\lambda}^2 (p^3 p' + pp'^3 + 3p^2 p'^2 - p^4 - p'^4) + O(\tilde{\lambda}^3) \right]. \quad (3.7)$$

One finds that

$$\langle kk' | \hat{S} | pp' \rangle = S(p', p) \delta_+(p, p', k, k'), \quad (3.8)$$

where the ‘kinematic’ factor $K(p, p')$ is included into the 2-body S -matrix $S(p', p)$.

¹² We are using a different normalization of creation operators than in [8] and thus some subsequent formulae differ by factors of 2π .

¹³ If one rescales the time coordinate and thus ω_p by $\tilde{\lambda}$ then $\omega_p = \sqrt{p^2 + m^2} - m^2$, $m^2 \equiv 1/\tilde{\lambda}$. This normalization corresponds to extracting one power of $\tilde{\lambda}$ from the spin-chain energy, so that the Heisenberg model energy does not have an overall $\tilde{\lambda}$ factor.

3.1. Leading-order tree-level term

Starting with the interaction term in (2.15) and (2.16) and computing the leading-tree-level contribution of the quartic vertex we obtain the following expression,

$$\begin{aligned}
 \begin{array}{c} \text{k} \nearrow \\ \text{p} \nearrow \\ \text{k}' \searrow \\ \text{p}' \searrow \end{array} &= -i \langle kk' | V_4 | pp' \rangle \\
 &= -i \left[\sqrt{1 + \tilde{\lambda}(p' - k')^2} + \sqrt{1 + \tilde{\lambda}(p' - k)^2} + \sqrt{1 + \tilde{\lambda}(p - k')^2} \right. \\
 &\quad \left. + \sqrt{1 + \tilde{\lambda}(p - k)^2} - \sqrt{1 + \tilde{\lambda}k'^2} - \sqrt{1 + \tilde{\lambda}k^2} - \sqrt{1 + \tilde{\lambda}p'^2} - \sqrt{1 + \tilde{\lambda}p^2} \right] \\
 &\quad - 2ia_1 \tilde{\lambda}^2 pp'kk' \\
 &\quad - \frac{i}{8} \tilde{\lambda}^3 pp'kk' [(2b_1 + b_2)(p + p')(k + k') - 2b_2(pp' + kk')] + O(\tilde{\lambda}^4), \quad (3.9)
 \end{aligned}$$

where for generality we kept the exact form of the square root terms in (2.15). Expanding in λ we get

$$\begin{aligned}
 -i \langle kk' | V_4 | pp' \rangle &= i\tilde{\lambda}(pp' + kk') + \frac{i\tilde{\lambda}^2}{8} [p^4 + p'^4 + k^4 + k'^4 - 4(k + k')(p^3 + p'^3) \\
 &\quad - 4(k^3 + k'^3)(p + p') + 6(k^2 + k'^2)(p^2 + p'^2) - 16a_1kk'pp'] \\
 &\quad + \frac{i\tilde{\lambda}^3}{4} \left\{ -\frac{1}{4}(p^6 + p'^6 + k^6 + k'^6) + 5(p^3 + p'^3)(k^3 + k'^3) \right. \\
 &\quad \left. + \frac{3}{2}[(k + k')(p^5 + p'^5) + (p + p')(k^5 + k'^5)] \right. \\
 &\quad \left. - \frac{15}{4}[(k^2 + k'^2)(p^4 + p'^4) + (p^2 + p'^2)(k^4 + k'^4)] \right. \\
 &\quad \left. - \frac{1}{2}(2b_1 + b_2)pp'kk'(p + p')(k + k') + b_2pp'kk'(pp' + kk') \right\} + O(\tilde{\lambda}^4). \quad (3.10)
 \end{aligned}$$

Taking into account the relations between p, p' and k, k' implied by momentum conservation (3.5), (3.6) we find the following contribution of the four-point vertex

$$\begin{aligned}
 -i \langle kk' | V_4 | pp' \rangle &= 2i\tilde{\lambda}pp' - i\tilde{\lambda}^2 \left[pp'(p^2 + p'^2) - \left(\frac{3}{2} - 2a_1 \right) p^2 p'^2 \right] \\
 &\quad + \frac{i}{4} \tilde{\lambda}^3 pp' \left[3(p^4 + p'^4) - \frac{1}{2}(15 + 2b_1 + b_2)pp'(p^2 + p'^2) \right. \\
 &\quad \left. + (10 - 2b_1 + b_2)p^2 p'^2 \right] + O(\tilde{\lambda}^4). \quad (3.11)
 \end{aligned}$$

Multiplying this by the kinematic factor in the delta-function (3.7) we obtain the leading terms in the tree-level 2-particle S -matrix

$$S(p', p) = 1 + S_{\text{tree}}(p', p) + \dots, \quad (3.12)$$

corresponding to the action (2.12), (2.15)

$$\begin{aligned}
 S_{\text{tree}}(p', p) &= \frac{2ipp'}{p - p'} \left\{ 1 + \tilde{\lambda}pp' \left(-\frac{1}{4} + a_1 \right) \right. \\
 &\quad \left. - \frac{1}{8} \tilde{\lambda}^2 pp' \left[(17 + 2b_1 + b_2)(p^2 + p'^2) + \frac{1}{2}(-9 + 2b_1 - b_2)pp' \right] + O(\tilde{\lambda}^3) \right\}. \quad (3.13)
 \end{aligned}$$

The comparison with the gauge Bethe ansatz allows one to fix the 2-loop coefficient a_1 and two of the three 3-loop coefficients— b_1 and b_2 . Indeed, for the values in (2.7) we get

$$S_{\text{tree}}^{(g)} = \frac{2ip p'}{p-p'} + \tilde{\lambda} \frac{ip^2 p'^2}{p-p'} - \tilde{\lambda}^2 \frac{ip^2 p'^2 (p^2 + p'^2 - pp')}{4(p-p')} + O(\tilde{\lambda}^3), \quad (3.14)$$

and this is the same S -matrix that comes out of the expansion of S_1 in (1.2) and (1.3) in small momenta $p_k = p'$, $p_j = p$ at each order in expansion in λ (see (4.3)). We thus generalize the result of [8] that the leading ‘1-loop’ term $\frac{2ip p'}{p-p'}$ in (3.13) which is the tree-level S -matrix of the standard LL model is the same as the small momentum expansion of the phase shift S_1 (1.2) of the Heisenberg spin-chain model to the next two orders in small λ expansion.

In the case of the LL model that originates from string theory and matches the predictions of the ‘string’ Bethe ansatz that includes the particular AFS [6] dressing phase θ in (1.2), the coefficients in equation (2.2) are given by (2.9) [31, 36] (i.e. a_1, b_1 are the same while b_2 is smaller by 1) one finds instead

$$S_{\text{tree}}^{(s)} = \frac{2ip p'}{p-p'} + \tilde{\lambda} \frac{ip^2 p'^2}{p-p'} - \tilde{\lambda}^2 \frac{ip^2 p'^2 (p^2 + p'^2)}{8(p-p')} + O(\tilde{\lambda}^3). \quad (3.15)$$

As expected, the ‘gauge’ (3.14) and ‘string’ (3.15) S -matrices differ starting at 3-loop order. Expression (3.15) follows indeed from the small λ expansion of the phase-shift factor of the AFS ansatz (see (4.13)).

We have also repeated the above computation including the λ^4 terms in (2.6) and checked that the resulting S -matrix for the values of ‘4-loop’ coefficients in (2.8) and (2.9) is again in agreement with the next-order $O(\tilde{\lambda}^3)$ term in (3.14) in the small momentum expansion of the BDS and AFS S -matrices (1.2), (1.3) given in (4.1) and (4.13).

3.2. 1-loop correction: order λ term

Let us now consider the 1-loop correction to the above tree-level S -matrix (3.13) following the same steps as in the leading-order ‘1-loop’ LL model case in [8]. One may try to compute subleading in small momentum expansion term at each order in small λ expansion. Here we shall consider the leading correction to the first two $O(\lambda^0)$ and $O(\lambda^1)$ terms in (3.13). While we will be expanding in λ , it is useful to keep the λ -dependent corrections to propagator before expanding in λ .¹⁴ At this order, the correction to the scattering matrix is (as mentioned before, kinematical constraints require that $p = k$ or $p = k'$; we consider explicitly only the diagram with $p = k, p' = k'$)

$$\begin{aligned} \text{Diagram} &= \int \frac{d\omega dq}{(2\pi)^2} D\left(\frac{p+p'}{2} + q\right) D\left(\frac{p+p'}{2} - q\right) \\ &\times \left[V\left(p, p', \frac{p+p'}{2} + q, \frac{p+p'}{2} - q\right) \right]^2 \\ D\left(\frac{p+p'}{2} + q\right) &= \frac{i}{\frac{1}{2}(\omega_p + \omega_{p'}) + \omega - \omega_{\frac{1}{2}(p+p')+q} + i\epsilon} \\ &= \frac{i}{\omega + \frac{\tilde{\lambda}}{4}(p^2 + p'^2) - \frac{\tilde{\lambda}^2}{16}(p^4 + p'^4) - \frac{\tilde{\lambda}}{2}\left(\frac{p+p'}{2} + q\right)^2 + \frac{\tilde{\lambda}^2}{8}\left(\frac{p+p'}{2} + q\right)^4 + O(\tilde{\lambda}^3) + i\epsilon} \end{aligned} \quad (3.16)$$

$$(3.17)$$

¹⁴ This takes into account the diagrams with extra insertions of the 4-derivative term in the kinetic part of (2.12) into the internal lines.

$$V = i\tilde{\lambda} \left[pp' + \frac{(p+p')^2}{4} - q^2 \right] + \frac{i\tilde{\lambda}^2}{8} [2q^4 + 3q^2(p-p')^2 - \frac{7}{8}(p^4 + p'^4) - \frac{9}{2}pp'(p^2 + p'^2) - \frac{21}{4}p^2p'^2] + O(\tilde{\lambda}^3) \quad (3.18)$$

where ω is the energy of the virtual particle with momentum q and we used (2.16) with $a_1 = \frac{3}{4}$. The integral over ω is easily done, and expanding the propagator in λ we find that the $O(\lambda^0)$ contribution to the scattering amplitude (3.8) is (ignoring δ_+ factor) [8]

$$-2 \frac{p^2 p'^2}{(p-p')^2}. \quad (3.19)$$

Here we divided by a symmetry factor of 2 and included the leading $\frac{\tilde{\lambda}^{-1}}{p-p'} \delta_+$ term from the kinematic factor (3.7).

At the next order in λ there are two contributions:

$$(i) = -\frac{i\tilde{\lambda}^2}{4} \int \frac{dq}{2\pi} \frac{pp' + \frac{(p+p')^2}{4} - q^2}{q^2 - \frac{(p-p')^2}{4}} \left[2q^4 + 3q^2(p-p')^2 - \frac{7}{8}(p^4 + p'^4) - \frac{9}{2}pp'(p^2 + p'^2) - \frac{21}{4}p^2p'^2 \right] \quad (3.20)$$

and

$$(ii) = \frac{i\tilde{\lambda}^2}{8} \int \frac{dq}{2\pi} \frac{(pp' + \frac{(p+p')^2}{4} - q^2)^2}{(q^2 - \frac{(p-p')^2}{4})^2} \left[p^4 + p'^4 - \left(\frac{p+p'}{2} + q \right)^4 - \left(\frac{p+p'}{2} - q \right)^4 \right]. \quad (3.21)$$

Evaluating the two integrals ignoring the power divergences by using the dimensional regularization prescription $\int dq q^\alpha = 0$ ($\alpha = 0, 1, 2, \dots$) as in [8], we obtain

$$(i) = 4\tilde{\lambda}^2 \frac{p^2 p'^2 (p^2 + p'^2)}{p-p'}, \quad (ii) = -2\tilde{\lambda}^2 \frac{p^2 p'^2 (p^2 + p'^2 + pp')}{p-p'}. \quad (3.22)$$

Adding them together (while dividing by a symmetry factor of 2) and including the delta-function factor in (3.7) we finally obtain the next to a leading-order term in the small momentum expansion of the 1-loop contribution to the S -matrix of the generalized LL model (2.12). Adding this 1-loop correction to the tree-level expression (3.19) we get

$$S_{\text{tree}+1\text{-loop}}(p, p') = -2 \frac{p^2 p'^2}{(p-p')^2} - 2\tilde{\lambda} \frac{p^3 p'^3}{(p-p')^2} + O(\tilde{\lambda}^2). \quad (3.23)$$

This expression agrees with the next-order term in the expansion in momenta of the S -matrix in the BDS and AFS Bethe ansätze in (4.1) and (4.13).

Furthermore, we can follow [8] and consider higher loop ‘bubble’ graphs and show that their contributions form a geometric series as in the case of the leading-order LL action. We shall postpone the details of this until section 5, where we will construct the all-order scattering matrix.

4. Small momentum expansion of S -matrix of BDS and AFS Bethe ansätze

Let us now determine explicitly the low-energy form of the spin-chain S -matrix in (1.2).

4.1. BDS case

Starting with the S -matrix of the BDS ansatz, i.e. with S_1 in (1.2) and (1.3)

$$S_{\text{BDS}}(p', p) = \frac{u(p') - u(p) + i}{u(p') - u(p) - i}, \quad (4.1)$$

$$u(p) = \frac{1}{2} \cot \frac{p}{2} \sqrt{1 + 4\tilde{\lambda} \sin^2 \frac{p}{2}}, \quad \tilde{\lambda} = \frac{\lambda}{(2\pi)^2}, \quad (4.2)$$

we may expand it in small momenta, and then also expand in λ

$$\begin{aligned} S_{\text{BDS}}(p', p) = & 1 + \frac{2ipp'}{p-p'} - 2 \frac{p^2 p'^2}{(p-p')^2} - \frac{ip^2 p'^2 (p^2 + 10pp' + p'^2)}{6(p-p')^3} + O(p^4) \\ & + \tilde{\lambda} \left[\frac{ip^2 p'^2}{p-p'} - \frac{2p^3 p'^3}{(p-p')^2} - \frac{ip^2 p'^2 (p^4 + 16p^2 p'^2 + p'^4)}{6(p-p')^3} + O(p^6) \right] \\ & - \tilde{\lambda}^2 \left[\frac{ip^2 p'^2 (p^2 - pp' + p'^2)}{4(p-p')} - \frac{p^3 p'^3}{2} \right. \\ & \left. - \frac{ip^2 p'^2 (p^6 - 3p^5 p' + 12p^4 p'^2 - 32p^3 p'^3 + 12p^2 p'^4 - 3pp'^5 + p'^6)}{16(p-p')^3} + O(p^8) \right] \\ & + \tilde{\lambda}^3 \left[\frac{ip^2 p'^2 (p^4 - p^3 p' + p^2 p'^2 - pp'^3 + p'^4)}{8(p-p')} + O(p^8) \right] + O(\tilde{\lambda}^4). \end{aligned} \quad (4.3)$$

To compare with the S -matrix of the effective Landau–Lifshitz model (2.1), (2.3) with the dispersion relation (3.2) we are to consider a particular resummation of part of the terms in expansion (4.3). We shall first keep only the leading term in small momentum expansion at each order in λ and then resum the series in λ . This will determine the tree-level S -matrix of the corresponding LL model. We can then also keep certain subleading terms in small momentum expansion that will combine into the geometric series (as in the leading-order LL model case in [8]). In this way we get

$$\begin{aligned} S_{\text{BDS}} \quad \rightarrow \quad \tilde{S}_{\text{BDS}}(p', p) = & 1 + \frac{2ipp'}{pe(p') - p'e(p)} - \frac{2p^2 p'^2}{[pe(p') - p'e(p)]^2} + \dots, \\ e(p) \equiv & \sqrt{1 + \tilde{\lambda} p^2} \end{aligned} \quad (4.4)$$

where $\frac{2ipp'}{pe(p') - p'e(p)}$ represents all leading terms in p at each order in λ .

The reason for this particular structure can be understood by noting that taking p small and keeping only the leading in p term in the expansion of the $\lambda \sin^2 \frac{p}{2}$ term in $u(p)$ in (4.1) gives

$$u(p) \rightarrow \frac{1}{p} \sqrt{1 + \tilde{\lambda} p^2}, \quad (4.5)$$

$$S_{\text{BDS}} = \frac{u(p') - u(p) + i}{u(p') - u(p) - i} \quad \rightarrow \quad \tilde{S}_{\text{BDS}} = \frac{1 + \frac{ipp'}{pe(p') - p'e(p)}}{1 - \frac{ipp'}{pe(p') - p'e(p)}}. \quad (4.6)$$

Then (4.4) follows upon expansion in small momenta with $e(p)$ kept fixed. This limit is thus formally equivalent to taking $p \rightarrow 0$ while keeping λp^2 fixed. As was mentioned in section 2.3, this limit is reminiscent of the BMN-type scaling limit with small p expansion corresponding to $1/J$ expansion (note that the structure of the LL action (2.1) is indeed consistent with this scaling limit).

While the first two terms (4.4) in the small momentum expansion of \tilde{S}_{BDS} (4.6) at fixed $e(p)$ are the same as in S_{BDS} , the higher order terms are different. This is clear already at the leading order in λ , i.e. at the level of the Heisenberg model versus the standard LL model. We have from (4.6) leads to

$$\tilde{S}_{\text{BDS}}(\lambda \rightarrow 0) = \frac{1 + \frac{ip'p'}{p-p'}}{1 - \frac{ip'p'}{p-p'}} = 1 + \frac{2ipp'}{p-p'} - \frac{2p^2p'^2}{(p-p')^2} - \frac{2ip^3p'^3}{(p-p')^3} + O(p^4), \quad (4.7)$$

where the order p^3 term is different from the similar one in the first line of (4.3) by

$$\frac{ip^2p'^2(p^2 + 10pp' + p'^2)}{6(p-p')^3} - \frac{2ip^3p'^3}{(p-p')^3} = \frac{ip^2p'^2}{6(p-p')}.$$
 (4.8)

This difference has its origin in the small momentum limit, which replaces $\frac{1}{2} \cot \frac{p}{2}$ with $\frac{1}{p}$ on the effective field-theory side. Interestingly, after extracting the kinematic factor (3.7) proportional to $\frac{1}{p-p'}$ this difference is represented by a local vertex which may thus be interpreted as a contribution of a local counterterm one may add to the leading-order LL action¹⁵.

4.2. AFS case

In the case of the ‘string’ Bethe ansatz of AFS [6] the scattering matrix (1.2) contains a particular dressing phase:

$$S_{\text{AFS}} = S_{\text{BDS}} e^{i\theta_{\text{AFS}}}, \quad \theta_{\text{AFS}} = 2 \sum_{r=2}^{\infty} \left(\frac{\tilde{\lambda}}{4} \right)^r [q_{r+1}(p)q_r(p') - q_{r+1}(p')q_r(p)], \quad (4.9)$$

where

$$q_r(p) = \frac{2 \sin((r-1)\frac{p}{2})}{r-1} \left(\frac{\sqrt{1 + 4\tilde{\lambda} \sin^2 \frac{p}{2}} - 1}{\tilde{\lambda} \sin \frac{p}{2}} \right)^{r-1}. \quad (4.10)$$

To match quantum string-theory results, the phase in (1.2) must receive modifications at subleading orders in strong coupling expansion [15].

The general expression for the phase is given by a double sum of the charges q_r [11, 15, 10] with coefficients having a non-trivial [15, 19–21] dependence on λ :

$$\theta(p', p; \lambda) = 2 \sum_{r=2}^{\infty} \sum_{s=r+1}^{\infty} c_{rs}(\lambda) \left(\frac{\tilde{\lambda}}{4} \right)^{\frac{r+s-1}{2}} [q_s(p)q_r(p') - q_s(p')q_r(p)]. \quad (4.11)$$

Here

$$c_{rs}(\lambda) = \delta_{s,r+1} + \frac{1}{\sqrt{\lambda}} a_{rs} + \frac{1}{(\sqrt{\lambda})^2} b_{rs} + \dots, \quad (4.12)$$

and $a_{rs} = \frac{4}{\pi} \frac{(r-1)(s-1)}{(r-1)^2 - (s-1)^2}$ for $r+s = \text{odd}$ and zero otherwise [19]. The relation between the general Bethe ansatz (1.1) with the phase (4.11) and the AFS ansatz should be understood as a statement that the coefficients c_{rs} at large λ reduce to $\delta_{s,r+1}$.¹⁶

¹⁵ One choice for such a counterterm is \tilde{n}^4 , which will have one less power of λ compared to the 2-loop term in (2.4). A more natural alternative would be the term \tilde{n}^2 coming out of subleading term in expansion of \sinh^2 term in (2.20). Strangely, the required coefficient of such a counterterm (5/6) happens to be different from that following from (2.20).

¹⁶ To say that AFS ansatz is a strong coupling limit of the general string ansatz is not precise as λ enters not only in c_{rs} but also in the expressions for u_j and q_r .

Let us first ignore the subleading terms in (4.12) and consider small momentum expansion of the AFS S -matrix (4.9) in the same way as we did above in the BDS case. Expanding in small momenta and then in λ we obtain

$$\begin{aligned}
S_{\text{AFS}}(p', p) = & 1 + \frac{2ipp'}{p-p'} - \frac{2p^2p'^2}{(p-p')^2} - \frac{ip^2p'^2(p^2+10pp'+p'^2)}{6(p-p')^3} + O(p^4) \\
& + \tilde{\lambda} \left[\frac{ip^2p'^2}{p-p'} - \frac{2p^3p'^3}{(p-p')^2} - \frac{ip^2p'^2(p^4+16p^2p'^2+p'^4)}{6(p-p')^3} + O(p^6) \right] \\
& - \tilde{\lambda}^2 \left[\frac{ip^2p'^2(p^2+p'^2)}{8(p-p')} - \frac{p^3p'^3}{4} + O(p^7) \right] \\
& + \tilde{\lambda}^3 \left[\frac{ip^2p'^2(p^4+p'^4)}{16(p-p')} + O(p^8) \right] - \tilde{\lambda}^4 \left[\frac{5ip^2p'^2(p^6+p'^6)}{128(p-p')} + O(p^{10}) \right] \\
& + \tilde{\lambda}^5 \left[\frac{7ip^2p'^2(p^8+p'^8)}{256(p-p')} + O(p^{12}) \right] + O(\tilde{\lambda}^6). \tag{4.13}
\end{aligned}$$

This expression is different from the expansion in (4.3) starting with the 3-loop $\tilde{\lambda}^2$ terms.

As in the BDS case, we may collect all the leading-order terms in small momentum at each order in λ , and then sum up the expansion in λ . The result may be again interpreted as a tree-level S -matrix of an effective field theory.

Given that the ‘string’ Bethe ansatz was constructed by starting with the strong-coupling region, here it may be more appropriate to view this low-energy limit as¹⁷

$$p \rightarrow 0, \quad \lambda p^2 = \text{fixed}, \tag{4.14}$$

i.e. as $p \rightarrow 0$ with $\lambda \sim p^{-2} \rightarrow \infty$. Since λ is then effectively taken to be large this suggests that in this limit quantum string $\frac{1}{\sqrt{\lambda}}$ corrections to the phase in (4.12) may be ignored. Indeed, as we shall find in section 7.2, this low-energy, strong coupling limit of the AFS S -matrix is in perfect agreement with the classical S -matrix of the LL-type model originating in a ‘non-relativistic’ limit from the string sigma model on $R \times S^3$.

Taking the limit $p \rightarrow 0$ with $\lambda p^2 = \text{fixed}$ in (4.9) and (4.10) as in (4.5) and (4.6) we get

$$q_r(p) \rightarrow p \left[\frac{e(p) - 1}{\frac{1}{2}\tilde{\lambda}p} \right]^{r-1}, \tag{4.15}$$

$$\theta_{\text{AFS}} \rightarrow \tilde{\theta}_{\text{AFS}} = (p'[e(p) - 1] - p[e(p') - 1]) \sum_{r=2}^{\infty} \left(\frac{[e(p) - 1][e(p') - 1]}{\tilde{\lambda}pp'} \right)^{r-1}. \tag{4.16}$$

Thus

$$\tilde{\theta}_{\text{AFS}} = (p'[e(p) - 1] - p[e(p') - 1]) \frac{[e(p) - 1][e(p') - 1]}{\tilde{\lambda}pp' - [e(p) - 1][e(p') - 1]} \tag{4.17}$$

so that

$$S_{\text{AFS}} \rightarrow \tilde{S}_{\text{AFS}} = \tilde{S}_{\text{BDS}} e^{i\tilde{\theta}_{\text{AFS}}}, \tag{4.18}$$

$$\tilde{S}_{\text{AFS}}(p', p) = \frac{1 + \frac{ipp'}{pe(p') - p'e(p)}}{1 - \frac{ipp'}{pe(p') - p'e(p)}} \exp \left[i \frac{(p'[e(p) - 1] - p[e(p') - 1])[e(p) - 1][e(p') - 1]}{\tilde{\lambda}pp' - [e(p) - 1][e(p') - 1]} \right]. \tag{4.19}$$

¹⁷ Here p stands, of course, for both p and p' .

Note that like $\frac{ip'p'}{pe(p')-p'e(p)}$ in (4.6) the phase θ_{AFS} scales linearly with momentum (at fixed λp^2) so that the leading term in the small momentum expansion is then

$$\tilde{S}_{\text{AFS}} = 1 + (\tilde{S}_{\text{AFS}})_{\text{tree}} + \dots, \quad (4.20)$$

$$(\tilde{S}_{\text{AFS}})_{\text{tree}} = \frac{2ipp'}{pe(p') - p'e(p)} + \frac{i(p'[e(p) - 1] - p[e(p') - 1])[e(p) - 1][e(p') - 1]}{\tilde{\lambda}pp' - [e(p) - 1][e(p') - 1]}. \quad (4.21)$$

The expansion of $(\tilde{S}_{\text{AFS}})_{\text{tree}}$ in powers of $\tilde{\lambda}$ then reproduces all the leading in small momentum terms at each order in λ in (4.13). An equivalent form of (4.21) is

$$(\tilde{S}_{\text{AFS}})_{\text{tree}} = \frac{2iF(p, p')}{pe(p') - p'e(p)}, \quad (4.22)$$

where

$$\begin{aligned} F(p, p') &= pp' + \frac{1}{2}[pe(p') - p'e(p)]\tilde{\theta}_{\text{AFS}} \\ &= \tilde{\lambda}^{-1}(\tilde{\lambda}pp' - [e(p) - 1][e(p') - 1])[1 + \frac{1}{4}(\tilde{\lambda}pp' - [e(p) - 1][e(p') - 1])]. \end{aligned} \quad (4.23)$$

By analogy with the BDS case one could expect that the expression in (4.19) may be possible to put into a ‘ratio’ form similar to (4.6), i.e. that the subleading terms in (4.20) should form geometric series

$$\tilde{S} = \frac{1 + \frac{iF(p, p')}{pe(p') - p'e(p)}}{1 - \frac{iF(p, p')}{pe(p') - p'e(p)}}. \quad (4.24)$$

This, however, does not follow from (4.19). Moreover, the expression in (4.19) or in (4.20) cannot be trusted beyond the leading term $(\tilde{S}_{\text{AFS}})_{\text{tree}}$ which scales as first power of momentum.

The reason is that the corrections to the phase (4.11), (4.12) which we ignored produce extra terms in the exponent in (4.19) that scale as quadratic and higher power of momenta. Indeed, in our low-energy limit (4.14) q_r in (4.15) scales as p^r and so the leading (AFS) term in the phase (4.11), (4.12) scales linearly with p . The subleading terms ignored in the AFS approximation (4.19) then scale as higher powers p^2, p^3, \dots and thus potentially contribute to the terms indicated by ellipsis in (4.20).

It could happen that for a special choice of the coefficients in (4.12) we could indeed end up with (4.24). It is possible to test this conjecture at the level of the first subleading term in (4.24) using the explicit value of the coefficients a_{rs} in (4.12). One finds (see appendix B) that the coefficient of the corresponding order p^2 correction to the phase depends on odd powers of our fixed parameter $\sqrt{\lambda}p$,

$$\delta\tilde{S}_{\text{AFS}} = -\frac{i}{3\pi}\tilde{\lambda}^{3/2}p^2p'(p - p') + \dots \quad (4.25)$$

i.e. it explicitly involves $\sqrt{\lambda}$, while F in (4.23) contains only integer powers of λ . This non-analyticity resulting from the first quantum correction to the phase has of course the same origin as that found in [15, 17]. It implies that the first subleading correction in (4.20) may not agree with the conjecture (4.24).¹⁸ Thinking of the LL action as a low-energy approximation to an effective quantum 2d string action (which contains string $\alpha' \sim \frac{1}{\sqrt{\lambda}}$ corrections) one may

¹⁸ It appears, however, that a more definitive statement requires knowing all higher order corrections. While the particular $\sqrt{\lambda}$ dependence of the first correction to θ leads to $\sqrt{\lambda}$ -dependent corrections to (4.19), a resummation of the full series may change this dependence.

be able to reproduce (4.25) and similar corrections starting with (2.12) and (2.16) where b_2 and other higher coefficients in (2.2) are actually functions of λ ,

$$b_2(\lambda) = b_{2s} + \frac{k_1}{\sqrt{\lambda}} + \dots, \quad (4.26)$$

interpolating between the weak coupling (2.7) and strong-coupling ($b_{2s} = -\frac{25}{2}$) (2.9) values [15, 36] (cf (3.13)).

The expression for $(\tilde{S}_{\text{AFS}})_{\text{tree}}$ (4.21) is more complicated than the corresponding one for $(\tilde{S}_{\text{BDS}})_{\text{tree}}$ (4.6). For that reason in the next section we shall use the BDS case to illustrate how to reconstruct an LL-type field theory model that reproduces such an S -matrix. We shall return to the AFS case in section 7.2 where we will show that (4.22) is precisely the S -matrix corresponding to the four-point vertex in the ‘non-relativistic’ limit of the classical string theory on $R \times S^3$.

5. All-order Landau–Lifshitz-type action corresponding to BDS S -matrix

We have seen that the generalized LL action (2.1)–(2.6) defined on an infinite line leads to the S -matrix which is the same as the leading terms in the small momentum expansion of the magnon S -matrix that enters the gauge-theory BDS or ‘string’ AFS Bethe ansätze. The matching depends on the proper choice of the coefficients of the interaction terms in the LL action: with the ‘gauge-theory’ choice (2.7), (2.8) we found that equations (3.14) and (3.23) match the respective terms in (4.3), while for the ‘string-theory’ choice (2.9) we found that the ‘3-loop’ term in (3.15) matches the corresponding term in (4.13). Equivalently, matching onto BDS or AFS S -matrix could be used to fix (part of) the coefficients ($a_1, b_1, b_2, c_5, c_7, \dots$) in the LL action.

We may then turn the problem around, i.e. follow the standard field theory logic and try to reconstruct the low-energy effective field theory which will be consistent with a particular small-momentum limit (4.4), (4.6) of the BDS S -matrix to *all* orders in expansion in λ . The same can be done also in the AFS case and thus may be important for understanding of how a similar S -matrix may be originating from quantum string theory.

The low-energy limit \tilde{S}_{BDS} (4.6) of the BDS S -matrix implies the dispersion relation (3.2), so that we shall assume that the effective action corresponding to \tilde{S}_{BDS} has the structure (2.12), where the interaction part V is to be determined.

Quite generally, a 2-body S -matrix fixes the on-shell value of the quartic vertex in V . The assumption that this field theory is integrable (implying factorization of the multi-particle S -matrix) determines the on-shell values of the interaction terms with higher number of fields in terms of the quartic one¹⁹. Some of the relations constraining them stem also from the $SO(3)$ symmetry of the \vec{n} -field LL action which is spontaneously broken to $U(1)$ symmetry of the action constructed from the on-shell vertex for the magnon fields ϕ, ϕ^* .

The above discussion suggests that the leading non-trivial term in (4.4)

$$(\tilde{S}_{\text{BDS}})_{\text{tree}} = \frac{2i p p'}{p e(p') - p' e(p)} \quad (5.1)$$

or in \tilde{S}_{BDS} in (4.6) may be interpreted as a tree-level field-theory S -matrix. A non-trivial consistency check that \tilde{S}_{BDS} can indeed be interpreted as a quantum S -matrix of an interacting field theory of LL type is that higher powers of $\frac{i p p'}{p e(p') - p' e(p)}$ (coming from the expansion of \tilde{S}_{BDS} in (4.6) in powers of momenta with $e(p)$ kept fixed) are the same as loop corrections

¹⁹ In particular, this means that the value of the coefficient b_3 in (2.5) is fixed by the values of b_1, b_2 , and similar relations should hold at higher orders in derivative expansion.

to the S -matrix of a two-dimensional field theory with the interaction vertex determined from (5.1). More precisely, these higher order terms should represent the contributions of bubble graphs with several insertions of this quartic vertex.

This is what we are going to show below, thus generalizing the relation [8] between the low-momentum (or ‘continuum’) limit of the Heisenberg chain S -matrix and the S -matrix of the quantum LL model to the BDS case, i.e. to all orders in λ . As a result, we will find an effective two-dimensional field theory behind the low-energy limit of the BDS S -matrix.

5.1. Tree-level four-point interaction vertex

Dividing (5.1) by the exact kinematic factor coming from the momentum conservation delta-function (3.7) we conclude that, up to the use of momentum conservation constraints, the exact on-shell four-point vertex should be given by

$$\mathcal{V}_{\text{on-shell}}(p, p') = i\tilde{\lambda} \frac{2pp'}{e(p)e(p')}. \quad (5.2)$$

Here the leading term in the expansion in λ is indeed consistent with (3.11). More precisely, the quartic term in the effective action (2.12) written in momentum representation with the four fields put on-shell has the form

$$\int dp dp' dk dk' \mathcal{V}(p, p'; k, k') K(p, p') \delta_+(p, p', k, k') a_p a_{p'} a_k^* a_{k'}^*, \quad (5.3)$$

where a_p is the Fourier transform of the on-shell field ϕ (3.1). The vertex $\mathcal{V}(p, p'; k, k')$ should be symmetric under the interchanges $p \leftrightarrow p', k \leftrightarrow k'$, and also under $(p, p') \leftrightarrow (k, k')$ to ensure the reality of the above expression. To extend the vertex off shell, we shall assume that the action has the same structure as at lowest orders (2.15) and (2.16), i.e. the interaction terms should involve only spatial derivatives. There are many possible off-shell extensions of (5.2) consistent with symmetries of (5.3); they will lead to actions differing only by field redefinitions. The simplest possible choice for the off-shell vertex is a $(p, p') \leftrightarrow (k, k')$ symmetrization of (5.2):

$$\mathcal{V}(p, p'; k, k') = \frac{i\tilde{\lambda}pp'}{e(p)e(p')} + \frac{i\tilde{\lambda}kk'}{e(k)e(k')}. \quad (5.4)$$

Expanding this in $\tilde{\lambda}$ we obtain

$$\mathcal{V} = i\tilde{\lambda}(pp' + kk') - \frac{i}{2}\tilde{\lambda}^2[pp'(p^2 + p'^2) + kk'(k^2 + k'^2)] + O(\tilde{\lambda}^3). \quad (5.5)$$

The $\tilde{\lambda}^2$ term here appears to be different from that in (3.10), though the two agree on-shell (i.e. the $\tilde{\lambda}^2$ terms in (3.11) and in (5.2) are the same). As we shall explain below, this is a reflection of a different choice of an off-shell extension.

In coordinate space (5.4) corresponds to the following term in the Lagrangian in (2.12) and (2.14):

$$V_4 = \frac{1}{4}\tilde{\lambda} \left[\left(\phi^* \frac{\partial_x}{\sqrt{1 - \tilde{\lambda}\partial_x^2}} \phi \right)^2 + \text{c.c.} \right]. \quad (5.6)$$

Again, the order $\tilde{\lambda}^2$ term in the expansion of (5.6) is different from its counterpart in (2.16). It may seem puzzling how the two actions can be related by a field redefinition given that the interaction terms do not involve time derivative while the kinetic term does. The way how

it works happens to be a peculiarity of first-order two-dimensional field theories. Consider a field redefinition $\phi \rightarrow \phi + \delta\phi$ with

$$\delta\phi = -\frac{3}{2}\tilde{\lambda}(\phi'^* \phi^2 + 2\phi\phi'\phi'^* + \phi^* \phi'^2). \quad (5.7)$$

This produces the following leading-order change in the Lagrangian in (2.12)

$$\delta\mathcal{L} = -i\left(\dot{\phi} - \frac{i}{2}\tilde{\lambda}\phi''\right)^* \delta\phi + \text{c.c.} + O(\phi^6). \quad (5.8)$$

The key observation is that the time-derivative term drops out since it can be written (using integration by parts) as a total derivative:

$$-i\dot{\phi}^* \delta\phi + \text{c.c.} = \frac{3i\tilde{\lambda}}{4} \frac{d}{dt} (\phi'^2 \phi^{*2} - \phi^2 \phi'^{*2}). \quad (5.9)$$

The remaining term $\frac{\tilde{\lambda}}{2}\phi''^* \delta\phi + \text{c.c.}$ gives an extra $\tilde{\lambda}^2$ contribution to the 4-vertex which is precisely the difference between (5.6) and (2.16). As expected, (5.8) and thus this remaining spatial derivative term vanishes on-shell, i.e. the two actions give the same tree-level S -matrix.

One may wonder which is an $SO(3)$ invariant action (2.1) for the unit vector \vec{n} which leads to the action (2.12) with the vertex given in (5.6). As follows from (2.15) and (3.9), the natural quadratic in \vec{n} term (2.3) in (2.1) produces a non-trivial contribution to the four-point amplitude that should be subtracted from (5.4) in order to determine the contribution that comes solely from the ‘interaction’ \vec{n}^4 terms in this action. The resulting ‘interaction’ vertex is then

$$\begin{aligned} \tilde{\mathcal{V}} &= \frac{i\tilde{\lambda}pp'}{e(p)e(p')} + \frac{i\tilde{\lambda}kk'}{e(k)e(k')} + i[e(p' - k') + e(p' - k) \\ &\quad + e(p - k') + e(p - k) - e(p) - e(p') - e(k) - e(k')]. \end{aligned} \quad (5.10)$$

It is not immediately clear how to write down explicitly the all-order \vec{n}^4 term which is consistent with such a vertex and also generalize the known terms in (2.4) and (2.6); one may need to use some field redefinitions to simplify it.

As already mentioned, the off-shell form (5.6) of the interaction vertex (5.4) is obviously not unique: in section 7.2 we shall present an alternative to (5.6) in which non-local factors are distributed symmetrically between the 4 fields in the vertex and which will be related to a simple scalar action with a 2-derivative kinetic term whose non-relativistic limit is the BDS LL model.

5.2. Loop corrections to field-theory S -matrix

Let us now consider quantum corrections to the 2-particle S -matrix using the vertex (5.4) and the same LL-model propagator as in (2.12) and (3.3), i.e. $D(\omega, p) = \frac{i}{\omega - e(p) + 1 + i\epsilon}$. Let us start with the 1-loop contribution, i.e. (3.16) now with the vertex (cf (3.18))

$$U(p, p'; q) \equiv \mathcal{V}\left(p, p', \frac{p+p'}{2} + q, \frac{p+p'}{2} - q\right) = \frac{i\tilde{\lambda}pp'}{e(p)e(p')} + \frac{i\tilde{\lambda}\left(\frac{(p+p')^2}{4} - q^2\right)}{e\left(\frac{p+p'}{2} - q\right)e\left(\frac{p+p'}{2} + q\right)}. \quad (5.11)$$

The energy (ω) integral in the 1-loop graph receives contribution from a single pole; it yields

$$I_1 = -i \int \frac{dq}{2\pi} \frac{[U(p, p'; q)]^2}{e(p) + e(p') - e\left(\frac{p+p'}{2} + q\right) - e\left(\frac{p+p'}{2} - q\right) + i\epsilon}. \quad (5.12)$$

Note that at large q the vertex $U(q)$ approaches a constant value while the denominator in (5.12) scales as $\sqrt{\lambda}q$ (the propagator of first-order theory scales linearly with inverse momentum)

implying the absence of power divergences but a potential presence of a logarithmic divergence (this is the same behaviour as, e.g., in the Thirring model). Indeed, expanding the integrand at large q we get²⁰

$$I_1 = -i \int \frac{dq}{2\pi q} \left[\frac{1}{2\sqrt{\tilde{\lambda}}} \left(\frac{\tilde{\lambda} p p'}{e(p)e(p')} - 1 \right)^2 + O\left(\frac{1}{q}\right) \right]. \quad (5.13)$$

This discussion was under an implicit assumption that λ was kept finite while one integrated over the momentum. If instead we first expand the integrand in λ and then do the integration over q separately at each order in λ we get power divergences but no logarithmic divergences. Similar result was previously found in [36] in the discussion of quantum corrections coming from quadratic in fluctuations terms in the generalized LL model (2.1), (2.3). To match the BDS S -matrix (which is essentially perturbative in λ) we need to adopt this second prescription and also to drop all power divergences (using, e.g., the dimensional regularization as in [8] as we already did at low order in λ in (3.16). Equivalently, this means that we should omit this ‘unphysical’ logarithmic divergence of the integral in (5.12) and (5.13).

To evaluate the finite part of the integral we note that, interestingly, the denominator of the integrand in (5.12) has two zeros, regardless the value of the coupling constant $\tilde{\lambda}$, at

$$q^2 = \frac{1}{4}(p - p')^2. \quad (5.14)$$

They correspond to simple poles. Only one of them contributes to the evaluation of the integral, independently of the choice of a contour. The residue at the relevant pole yields

$$I_{1\text{pole}} = \left[U\left(p, p'; \frac{p - p'}{2}\right) \right]^2 \frac{\tilde{\lambda} e(p)e(p')}{pe(p') - p'e(p)}, \quad (5.15)$$

where from (5.11) and (5.2) we get

$$U\left(p, p'; \frac{p - p'}{2}\right) = \frac{2\tilde{\lambda} i p p'}{e(p)e(p')} = \mathcal{V}_{\text{on-shell}}(p, p'). \quad (5.16)$$

In addition to the contribution of the residue at the pole at finite distance, there is also a contribution from the contour at infinity in the complex q plane. To evaluate it let us set $q = R e^{i\psi}$ and pull out a factor of $\frac{1}{e(q)}$, while expanding the rest of the expression in large R . We obtain then for the contour integral

$$\tilde{\lambda}^2 \int_{\pi}^{2\pi} \frac{d\psi}{2\pi} \frac{R e^{i\psi}}{\sqrt{1 + \tilde{\lambda} R^2 e^{2i\psi}}} \left[\frac{1}{2} \left(\frac{\tilde{\lambda} p p'}{e(p)e(p')} - 1 \right)^2 + O\left(\frac{f(\psi)}{R}\right) \right]. \quad (5.17)$$

The $O\left(\frac{f(\psi)}{R}\right)$ terms contain convergent integrals, so that after taking the $R \rightarrow \infty$ limit they vanish. The remaining term which should be formally added to the pole contribution in (5.15) gives the logarithmically divergent contribution, i.e.

$$I_1 = \left[U\left(p, p'; \frac{p - p'}{2}\right) \right]^2 \frac{\tilde{\lambda}^{-1} e(p)e(p')}{pe(p') - p'e(p)} + \frac{i}{2\pi\sqrt{\tilde{\lambda}}} \left(\frac{\tilde{\lambda} p p'}{e(p)e(p')} - 1 \right)^2 \log(2R\sqrt{\tilde{\lambda}}). \quad (5.18)$$

As we have already discussed above, this logarithmic divergence, i.e. the contribution of the contour integral, should be omitted assuming that the BDS S -matrix and thus the corresponding LL model should be understood perturbatively in λ .

²⁰ Somewhat surprisingly, the coefficient of the logarithmic divergence happens to have a non-analytic dependence on λ (cf [36]).



Figure 1.

Dividing by the symmetry factor 2, and multiplying by the kinematic factor in (3.7) we thus find the following 1-loop contribution to the 2-particle field-theory S -matrix

$$\frac{1}{2}(I_1)_{\text{fin}} \frac{\tilde{\lambda}^{-1}e(p)e(p')}{pe(p') - p'e(p)} = -\frac{2p^2p'^2}{[pe(p') - p'e(p)]^2}. \quad (5.19)$$

This matches exactly the second term in (4.4), i.e. the term in the small momentum expansion of \tilde{S}_{BDS} in (4.6).

One can extend this computation to higher loop bubble graphs (figure f1) as in [8]. Omitting again logarithmic divergences (or all power divergences if one first expands in λ as discussed above) we finish with the following n -loop contribution to the 2-particle scattering

$$(I_n)_{\text{fin}} = \left[U \left(p, p'; \frac{p - p'}{2} \right) \right]^{n+1} \left[\frac{\tilde{\lambda}^{-1}e(p)e(p')}{pe(p') - p'e(p)} \right]^n. \quad (5.20)$$

Dividing by the symmetry factor 2^n and multiplying by the kinematic factor from (3.7) we finish with the following generalization of (5.19)

$$\frac{1}{2^n}(I_n)_{\text{fin}} \frac{\tilde{\lambda}^{-1}e(p)e(p')}{pe(p') - p'e(p)} = 2 \left[\frac{ipp'}{pe(p') - p'e(p)} \right]^{n+1}. \quad (5.21)$$

Summing up all these bubble diagram contributions gives (as at leading order in λ [8]) a simple geometric series and thus we finish with the following field-theory S -matrix:

$$S_{\text{LL}}(p', p) = \frac{pe(p') - p'e(p) + ipp'}{pe(p') - p'e(p) - ipp'}. \quad (5.22)$$

This is indeed exactly the same as the low-energy limit of the BDS S -matrix, i.e. \tilde{S}_{BDS} in (4.6). It may be viewed as a generalization to all orders in λ of the standard ('1-loop') Landau–Lifshitz model S -matrix obtained in [8].

It is worth stressing that it was not *a priori* clear that the result of this calculation should yield (the low-energy limit of) the BDS S -matrix. This conclusion rests on a number of details, in particular, on the structure of the quadratic term as well as of the quartic vertex in the LL action, making the agreement non-trivial. In appendix A, we shall generalize the construction of the quartic vertex in the BDS-related LL action to the $SU(1|1)$ and $SL(2)$ sectors.

The above discussion may be repeated also in the AFS case by starting with the 4-vertex consistent with (4.22) which we shall explicitly determine in section 7.2. Loop corrections to the S -matrix of such an LL model produce again a geometric series combining into (4.24).²¹ However, expression (4.24) does not naturally follow from the AFS S -matrix (4.19) (see comments at the end of section 4); the important issue of the relation between the low-energy limit of the exact string S -matrix and the quantum S -matrix (4.24) of the LL model with tree-level AFS vertex remains to be clarified.

²¹ Since the vertex corresponding to (4.22) scales with momentum in the same way as in the BDS case here again we shall get a formal logarithmic divergence of 1-loop integral which should be discarded in the LL framework. Similar divergences should be automatically cancelling only in the full superstring calculation where both positive and negative-energy modes will be propagating in loops and also contributions of other bosons and fermions will be included.

6. Comments on larger sectors

In the previous sections, we have constructed a field theory whose loop expansion reproduces the BDS S -matrix. A natural question is whether a similar field theory exists for larger sectors. Here we shall make few comments on the case of sectors including the $SU(2)$ sector.

6.1. S -matrix of the Landau–Lifshitz model for the $SU(1|2)$ sector

Let us consider, for example, the $SU(1|2)$ sector (containing gauge-theory operators built out of two chiral scalars and one component of gaugino [44]) where the leading-order LL Lagrangian is given by [45, 46] (cf (2.1) and (2.3))

$$\mathcal{L} = -iU_i^* \partial_0 U_i - i\psi^* D_0 \psi - \frac{\tilde{\lambda}}{2} [(1 - \psi^* \psi) |D_1 U_i|^2 + D_1^* \psi^* D_1 \psi], \quad (6.1)$$

where $D_a = \partial_a - iC_a$, $C_a = -iU_i^* \partial_a U_i$ and $|U_1|^2 + |U_2|^2 = 1$. Expanding near the vacuum configuration $U_1 = 1, U_2 = 0$ (i.e. $\vec{n} = (0, 0, 1)$ for $\vec{n} = U^* \vec{\sigma} U$) and $\psi = 0$ and rescaling the spatial coordinate by J as in (2.10) we obtain the following action to quartic order in the fluctuation fields ϕ and ψ which generalizes (2.12) to the presence of a complex fermion field:

$$\begin{aligned} \mathcal{S} &= \int dt \int_0^J dx \left[\phi^* \left(i\partial_t + \frac{1}{2} \tilde{\lambda} \partial_x^2 \right) \phi - \psi^* \left(i\partial_t - \frac{1}{2} \tilde{\lambda} \partial_x^2 \right) \psi - V_4(\phi, \phi^*, \psi, \psi^*) \right] \quad (6.2) \\ V_4 &= \frac{\tilde{\lambda}}{4} [(\phi^{*2} \phi'^2 + \text{c.c.}) - 2\psi^* \psi |\phi'|^2 + [(\phi' \phi^* - \phi \phi'^*) \psi'^* \psi + \text{c.c.}]] + \frac{i}{2} (\dot{\phi} \phi^* - \phi \dot{\phi}^*) \psi^* \psi. \end{aligned} \quad (6.3)$$

Here we followed the previously used notation for ϕ, ϕ^* in (2.11) and the notation for the fermions in [45, 46]; to make the signs in the respective kinetic terms in (6.2) the same it is sufficient to interchange ϕ with ϕ^* or ψ with $\bar{\psi}$.

The time-derivative-dependent interaction term in (6.3) can be converted into a spatial derivative one by a field redefinition. More precisely, following the logic outlined in equation (5.8) combined with the transformation $\phi \rightarrow \phi + \frac{1}{2} \phi \psi \bar{\psi}$ replaces the time derivatives in (6.3) with spatial derivatives. The resulting V_4 can be simplified to

$$V_4 = \frac{\tilde{\lambda}}{4} [(\phi^{*2} \phi'^2 + \text{c.c.}) + 2(\psi^* \psi' \phi \phi'^* + \text{c.c.})]. \quad (6.4)$$

The solutions to the free fermion equations of motion may be chosen as

$$\psi^*(x, t) = \int \frac{dp}{\sqrt{2\pi}} b_p e^{-i\omega_p t + i p x}, \quad \psi(x, t) = \int \frac{dp}{\sqrt{2\pi}} b_p^* e^{i\omega_p t - i p x} \quad (6.5)$$

with ω_p being the same as in (3.2) and $\{b_p, b_{p'}^*\} = \delta(p - p')$ (this choice assures the positivity of energy). Then the fermionic propagator is the same as the bosonic one (3.3).²²

In addition to the bosonic vertex (2.16) that we had in the $SU(2)$ sector, now we have also a 2 boson–2 fermion vertex shown in figure 2(a), where the dashed line denotes the fermion.

Suppose we are interested in the bosonic sector of the S -matrix where the *in* and *out* particles are bosons as in the $SU(2)$ case. The tree-level scattering matrix is then the same as in the $SU(2)$ sector, while at 1-loop level we could get an additional contribution from the fermionic loop in figure 2(b). However, this contribution *vanishes* since, like the bosonic propagator,

²² As already mentioned, an alternative way to make the analogy with the free bosonic theory manifest is to switch $\psi^* \leftrightarrow \psi$.

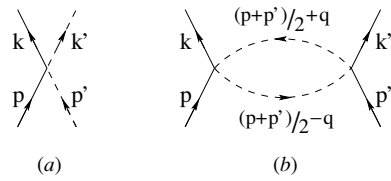


Figure 2. (a) Elastic scattering of a bosonic and fermionic magnon; (b) potential fermionic correction to the scattering of bosonic magnons.

the fermionic propagator is also retarded (see also [8]). This argument extends also to higher loop contributions.

We conclude that the bosonic sector of the S -matrix of the $SU(1|2)$ LL model is exactly the same as that of the $SU(2)$ LL model. This appears to be in agreement with the structure of the corresponding Bethe ansatz S -matrix.

6.2. Absence of mixing of magnon S -matrices

The decoupling observed in the previous subsection for the all-loop scattering of bosons and fermions is not restricted to the $SU(1|2)$ sector. It is possible to see that in the context of the LL-type models, all larger sectors containing $SU(2)$ have the same property: the quantum LL S -matrix with external states from the $SU(2)$ sector is simply that of the quantum $SU(2)$ LL model.

There are two essential ingredients which lead to this type of decoupling. First, the scattering of magnons is flavour-diagonal. In the case of either the ‘gauge’ or ‘string’ Bethe ansatz the magnons are associated with simple roots of $PSU(2, 2|4)$ and scatter following its Dynkin diagram [5]. As a consequence, a field theory realizing the magnon scattering should have a very particular form of the 4-field interaction terms. Since the magnons are associated with the nodes of the Dynkin diagram, we may assign to them Abelian conserved charges. Each term in the magnon Lagrangian is neutral with respect to these charges; thus each term must contain an equal number of magnon fields and their complex conjugates (in particular, cubic vertices are prohibited).

Second, the magnons around the ferromagnetic ground state of any spin chain are non-relativistic. Therefore, it is always possible to choose the vacuum of the field theory describing them in such a way that it is annihilated by holomorphic fields. Hence all propagators are retarded: $\langle \phi^*(t, x)\phi(t', y) \rangle|_{t' < t} = 0$.

It follows then that at the tree-level there is no term in the effective field-theory Lagrangian which corresponds to annihilation of some type of magnon and pair-production of a different kind of magnon. Instead, all terms describe elastic scattering (as in figure 2(a), showing the scattering of a boson and a fermion). Also, loop contributions containing different sorts of magnons than those on external legs vanish. The conclusion is that magnon scattering in each unit rank sector does not receive corrections from other sectors.

It is worth mentioning that if the magnons were charged under more than one Cartan generator, then cubic vertices would be allowed in the effective field-theory action, and, in spite of the propagators being retarded, there could exist, e.g., a non-trivial fermion contribution to the scattering of bosons.

This pattern is obviously different from what is found in analogous string-theory second-derivative sigma model computations, where loop diagrams involving all states provide non-trivial contributions to diagrams with external states from the $SU(2)$ sector (and, in fact, to all

unit-rank sectors). These contributions are crucial, in particular, for the cancellation of 2d UV divergences and presumably for obtaining the complete dressing phase θ .

7. The relation to string theory

Our motivation for reconstructing the field theory whose scattering matrix reproduces the asymptotic S -matrix of the spin chain (1.1) is the expected close relation of this S -matrix with string theory in $AdS_5 \times S^5$. The field-theory action we discussed above is a Landau–Lifshitz-type non-relativistic action which is first order in time derivative. At the same time, the $AdS_5 \times S^5$ string action expanded near the point-like string moving along the S^5 geodesic has a two-dimensional relativistically invariant kinetic term [49, 25] and the interaction terms which are not relativistically invariant [12, 13, 50, 51].

To relate such a second-order action for ‘BMN magnons’ to first-order LL-type action (2.12) for ‘spin-chain magnons’ it is necessary to eliminate half of the modes in the former action—the negative-energy modes. It is also necessary to eliminate time-derivative terms from the interaction part.

A general systematic procedure (not assuming the expansion of \vec{n} near the $(0, 0, 1)$ vacuum) for relating the classical string action on $R \times S^3$ to generalized LL action (2.1)–(2.5) was presented in [31]. It was based on the ‘fast-string’ expansion (i.e. it assumed that the time derivatives of ‘transverse’ string profile \vec{n} are small compared to spatial ones) and on performing field redefinitions to eliminate time derivatives from the interaction terms. This determined the exact quadratic term in the action and also few leading coefficients a_1, b_1, b_2, b_3 in (2.4) and (2.5) (see (2.7) and (2.9)). The ‘string’ values of these coefficients are indeed consistent with the string AFS-type Bethe ansatz [36].

While directly extending the approach of [31] to determine the coefficients of higher order terms in the corresponding effective LL action appears to be complicated, expanding near the vacuum $\vec{n} = (0, 0, 1)$ and concentrating only on the quartic interaction vertex one is able to fix its exact form as we shall do in section 7.2.

We shall find that it matches exactly the vertex extracted from the tree-level part of the low-energy limit of the AFS S -matrix (4.22). This may be expected, given that the AFS Bethe ansatz was obtained by discretizing [6] the classical $R \times S^3$ string Bethe equations of [32], but it gives a hope of more direct understanding of the correspondence between quantum string corrections and the structure of string S -matrix (cf also [9]).

Below we will first explain how one can relate the LL model with first-order kinetic term and the vertex of the type of (5.6) and (5.2) or its AFS analogue to an interacting 2d action with a relativistic kinetic term. We shall then see that there is indeed a close connection between the string sigma model action on $R \times S^3$ and the LL action of the type (2.12), (5.6), which generalizes the leading-order classical correspondence described in [30, 31] to all orders in λ (but only for quartic interaction terms).

7.1. A model scalar field theory

Let us start with an illustrative example of an effective massive two-derivative field theory obtained, for example, by expanding the string sigma model around some semiclassical solution²³. Such an effective action will naturally have a kinetic term with two space and two

²³ It may have two possible origins. First, we may think of integrating out all string fields except few which will appear in the effective action. Such an action would necessarily exhibit divergences which should disappear once quantum effects of the modes present in this action are also included (the total quantum string theory is expected to be finite). This action should not be interpreted in the Wilsonian sense, as the remaining fields may be equally

time derivatives and thus, unlike the LL-type models, will contain both positive and negative-energy modes. As already mentioned, the latter should be eliminated in order to bring it to the LL form. Below we will describe how this can be done and as a result reproduce some characteristic features of the LL actions discussed in the previous sections.

One such feature is the occurrence of inverse powers of $e(p) = \sqrt{1 + \tilde{\lambda} p^2}$ in the quartic vertex. While the precise structure of the vertex depends on the details of the effective action, the presence of inverse powers of $e(p)$ appears to be directly related to the elimination of the negative-energy modes.

Let us start with a generic complex scalar Lagrangian,

$$L = -\phi^*(\partial_t^2 - \partial_x^2 + m^2)\phi - \hat{V}_4(\partial^{(i)})\phi^*(z_1)\phi^*(z_2)\phi(z_3)\phi(z_4) + \dots \quad (7.1)$$

Here the ellipsis denote terms involving more than four fields (which are presumably fixed by the integrability of the theory). $\partial^{(i)}$ collectively denote space and time derivatives acting on the field at position $z_i = (t_i, x_i)$.²⁴

Given action (7.1) we may compute the corresponding tree-level S -matrix by solving the corresponding classical equations with the free in-field boundary condition. The free field can be decomposed into the positive- and negative-energy modes

$$\phi = \phi_+ + \phi_-, \quad (7.2)$$

which will then enter also the nonlinear solution and thus the resulting S -matrix. Since we will be interested only in the 2-body S -matrix, i.e. in the quartic vertex, we may formally use the decomposition (7.2) directly in the action.

The kinetic operator can be factorized as

$$\partial_t^2 - \partial_x^2 + m^2 = -D_+ D_- \quad (7.3)$$

$$D_{\pm}(\partial) \equiv i\partial_t \mp e(i\partial_x), \quad e(i\partial_x) \equiv \sqrt{m^2 - \partial_x^2}, \quad (7.4)$$

so that $D_+\phi_- = O(\phi^3)$, $D_-\phi_+ = O(\phi^3)$. Suppose we consider diagrams where only ϕ_+ and ϕ_+^* appear on external lines and ask which field theory with first-order kinetic term D_- would reproduce the same S -matrix. To arrive at such action from (7.1) we may supplement the decomposition (7.2) by a field redefinition

$$\hat{\phi}_{\pm} = \sqrt{D_{\mp}(\partial)}\phi_{\pm}. \quad (7.5)$$

Then (7.1) expressed in terms of $\hat{\phi}_{\pm}$ becomes

$$L = \hat{\phi}_+^*(i\partial_t - \sqrt{m^2 - \partial_x^2})\hat{\phi}_+ - \frac{\hat{V}_4(\partial^{(i)})}{\sqrt{D_-(\partial^{(1)})D_-(\partial^{(2)})D_-(\partial^{(3)})D_-(\partial^{(4)})}} \\ \times \hat{\phi}_+^*(z_1)\hat{\phi}_+^*(z_2)\hat{\phi}_+(z_3)\hat{\phi}_+(z_4) + \text{terms containing } \hat{\phi}_-, \hat{\phi}_-^*. \quad (7.6)$$

At the tree-level ignoring the dependence on $\hat{\phi}_-, \hat{\phi}_-^*$ means consistently truncating the S -matrix to the sector of the positive-energy modes. The first two terms in (7.5) then give a

massive as those which have been integrated out at the first stage. Rather, this may be viewed as a way to exactly account for the quantum effects of some of the fields while treating others as classical. Alternatively, we may think of this effective action in the usual 1PI sense. Then all fields are allowed to propagate in the loops so that this effective action should be free of 2d UV divergences. As usual, the exact quantum 2d S -matrix is then the tree-level S -matrix of such an effective action.

²⁴ While generically non-local, the quartic interaction term typically can be expanded in series of local operators of increasing dimension. An exception is the case in which some of the world-sheet fields which have been eliminated are massless around the chosen background.

first-order action that resembles the LL actions discussed above (cf (2.12)). However, there is an obvious difference in that the interaction term may contain time derivatives.

To address this issue let us note that the truncation to positive energy modes implicitly assumes that the resulting action is to be used in the low-energy regime ($\omega \ll m$), where the excitations of the field ϕ can be thought of as non-relativistic. It is therefore reasonable to expand the D_- factors in the interaction vertex in (7.6) in $\frac{\partial_t}{m}$. Then we can further eliminate the time-derivative-dependent terms in the vertex using field redefinitions (as in the relation between the string sigma model and the LL action in [31]), i.e. using that on the free equations of motion $i\partial_t \hat{\phi}_+ = e(i\partial_x) \hat{\phi}_+$. Equivalently, we may just replace $i\partial_t$ by $\sqrt{m^2 - \partial_x^2}$ in the quartic interaction term. We then finish with the following effective Lagrangian for $\hat{\phi}_+$

$$L \simeq \hat{\phi}_+^* [i\partial_t - e(i\partial_x)] \hat{\phi}_+ - V_4, \quad (7.7)$$

$$V_4 = \frac{\hat{V}_4[-ie(i\partial_x^{(i)}), \partial_x^{(i)})]}{4\sqrt{e(i\partial_x^{(1)})e(i\partial_x^{(2)})e(i\partial_x^{(3)})e(i\partial_x^{(4)})}} \hat{\phi}_+^*(z_1)\hat{\phi}_+^*(z_2)\hat{\phi}_+(z_3)\hat{\phi}_+(z_4). \quad (7.8)$$

Comparing this to the on-shell vertex (5.2) written in an equivalent symmetric form

$$\mathcal{V}(p, p'; k, k') = i\bar{\lambda} \frac{pp' + kk'}{\sqrt{e(p)e(p')e(k)e(k')}}, \quad (7.9)$$

we observe its close similarity with the vertex (7.7) extracted from the BDS S -matrix provided we choose V_4 in a remarkably simple local form

$$\hat{V}_4(\partial^{(i)}) \propto [\partial_x^{(1)}\partial_x^{(2)} + \partial_x^{(3)}\partial_x^{(4)}] \prod_{i=1}^4 \delta^{(2)}(z_i - z). \quad (7.10)$$

There is still a difference in the structure of the kinetic terms in (7.7) and in (2.12): apart from the replacement of m^2 by $\tilde{\lambda}^{-1}$ (and a rescaling of t) here we are missing the subtraction of -1 in the dispersion relation (3.2). This can be easily fixed by applying a field redefinition $\phi = e^{-imt} \tilde{\phi}$, where $\tilde{\phi}_+$ will now be a ‘slow’ field. This mimicks the ‘fast string’ expansion (based on isolating the fast angle variable) done in relating the LL action to string theory action in [31] (see also [47, 48]). Then

$$-\phi^*(\partial_t^2 - \partial_x^2 + m^2)\phi = 2im\tilde{\phi}^*\partial_t\tilde{\phi} - \tilde{\phi}^*(\partial_t^2 - \partial_x^2)\tilde{\phi} \quad (7.11)$$

$$\hat{\phi}_+^*(i\partial_t - \sqrt{m^2 - \partial_x^2})\hat{\phi}_+ = \tilde{\phi}_+^*[i\partial_t - (\sqrt{m^2 - \partial_x^2} - m)]\tilde{\phi}_+ \quad (7.12)$$

and so the transformation from the relativistic to non-relativistic theory can be viewed as a standard non-relativistic expansion.

7.2. Relation between string sigma model on $R \times S^3$ and the AFS S -matrix

Let us now turn to string theory and explain how the above action appears in the context of the discussion of [31]. There one started with the classical string action on $R \times S^3$ with the metric $ds^2 = -dt^2 + [d\alpha + C(n)]^2 + d\vec{n} d\vec{n}$, performed 2d duality $\alpha \rightarrow \tilde{\alpha}$, and then fixed the ‘uniform’ gauge: $t = \tau$, $\tilde{\alpha} = \frac{J}{\sqrt{\lambda}}\sigma$, i.e. $p_\alpha = \frac{J}{\sqrt{\lambda}} = \text{const.}$ ²⁵ The resulting action then takes

²⁵ The choice of the isometry direction α in fixing the uniform gauge corresponds to a particular choice of a charge that is assumed to be distributed homogeneously along the string to match the spin-chain picture [31]. In the gauge used in [31, 48] that isometry direction corresponded to the total spin $J = J_1 + J_2$ in the $SU(2)$ sector, while in the uniform gauge used in [58] the corresponding charge was single spin component J_1 (i.e. α was the angle in one of the three rotation planes).

the form (2.1); after the redefinition (2.10) of the world-sheet coordinate $\sigma \rightarrow x = \frac{J}{2\pi}\sigma$ the string Lagrangian [31] takes the J -independent form ($S = \int dt \int_0^J dx L$)

$$\mathcal{L} = C_t - \sqrt{\left[1 - \frac{1}{4}(\partial_t \vec{n})^2\right] \left[1 + \frac{\tilde{\lambda}}{4}(\partial_x \vec{n})^2\right] + \frac{\tilde{\lambda}}{16}(\partial_t \vec{n} \cdot \partial_x \vec{n})^2}. \quad (7.13)$$

Expanding (7.13) near $\vec{n} = (0, 0, 1)$ and using (2.11) we get for the terms quartic in fluctuation field (cf (2.12) and (2.16))

$$L = i\phi^* \partial_t \phi - \frac{1}{2}\phi^* (\partial_t^2 - \tilde{\lambda} \partial_x^2) \phi + \frac{1}{4}[\phi^{*2}(\phi^2 - \tilde{\lambda} \phi'^2) + \text{c.c.}] + \frac{1}{8}(\phi^{*2} - \tilde{\lambda} \phi'^{*2})(\phi^2 - \tilde{\lambda} \phi'^2) + O(\phi^6). \quad (7.14)$$

We can now apply to this action the procedure from the previous subsection to read off the quartic vertex in the corresponding ‘non-relativistic’ action.

The first step is to replace the time derivatives in the quartic interaction term in (7.14) with their expression following from the free equations of motion (the result is the same as doing field redefinitions and ignoring higher than quartic terms):

$$\partial_t \phi \rightarrow -i[e(i\partial_x) - 1]\phi, \quad e(i\partial_x) \equiv \sqrt{1 - \tilde{\lambda} \partial_x^2}. \quad (7.15)$$

The resulting quartic vertex (5.3) is then easily found from (7.14) in momentum representation:

$$\mathcal{V}(p, p'; k, k') = i \frac{\tilde{V}_4(p, p', k, k')}{\sqrt{e(p)e(p')e(k)e(k')}}, \quad (7.16)$$

$$\tilde{V}_4 = \tilde{\lambda}(pp' + kk') - [e(p) - 1][e(p') - 1] - [e(k) - 1][e(k') - 1] + \frac{1}{2}(\tilde{\lambda}pp' - [e(p) - 1][e(p') - 1])(\tilde{\lambda}kk' - [e(k) - 1][e(k') - 1]). \quad (7.17)$$

Upon multiplication of this by the energy–momentum delta-function in (3.5) and (3.7) (allowing us to set k, k' to p, p' or p', p) we then reproduce precisely the ‘tree-level’ part of the low-energy AFS S -matrix in (4.22) and (4.23):

$$\mathcal{V}(p, p', k, k') \frac{\tilde{\lambda}^{-1} e(p)e(p')}{pe(p') - p'e(p)} \delta_+(p, p', k, k') = (S_{\text{string}}^{\text{SU}(2)})_{\text{tree}} \delta_+(p, p', k, k') \quad (7.18)$$

$$(S_{\text{string}}^{\text{SU}(2)})_{\text{tree}} = \frac{2iF(p, p')}{pe(p') - p'e(p)}. \quad (7.19)$$

Following the procedure of the previous subsection, the resulting non-relativistic effective Lagrangian corresponding to (7.14) is (cf (7.7) and (2.12))

$$L = \phi^* [i\partial_t - (\sqrt{1 - \tilde{\lambda} \partial_x^2} - 1)] \phi - V_4(\phi) + O(\phi^6), \quad (7.20)$$

$$V_4 = \frac{1}{4} \left\{ \left(\frac{1}{\sqrt{e(i\partial_x)}} \phi^* \right)^2 \left[\left(\frac{e(i\partial_x) - 1}{\sqrt{e(i\partial_x)}} \phi \right)^2 + \tilde{\lambda} \left(\frac{\partial_x}{\sqrt{e(i\partial_x)}} \phi \right)^2 \right] + \text{c.c.} \right\} - \frac{1}{8} \left[\left(\frac{e(i\partial_x) - 1}{\sqrt{e(i\partial_x)}} \phi^* \right)^2 + \tilde{\lambda} \left(\frac{\partial_x}{\sqrt{e(i\partial_x)}} \phi^* \right)^2 \right] \left[\left(\frac{e(i\partial_x) - 1}{\sqrt{e(i\partial_x)}} \phi \right)^2 + \tilde{\lambda} \left(\frac{\partial_x}{\sqrt{e(i\partial_x)}} \phi \right)^2 \right]. \quad (7.21)$$

Expanding (7.20) in powers of spatial derivatives one can check that it agrees (modulo field redefinitions like that below (5.6) with the leading terms in the LL vertex (2.16) for the ‘string’ value of the coefficient b_2 in (2.9).

V_4 in (7.21) is the ‘string’ or AFS analogue of the exact off-shell BDS field-theory vertex (5.6). To make the analogy with (7.21) more explicit we can represent the on-shell BDS vertex (5.4) in a more symmetric form (equivalent on-shell to (5.6))

$$(V_4)_{\text{BDS}} = \frac{\tilde{\lambda}}{4} \left[\left(\frac{1}{\sqrt{e(i\partial_x)}} \phi^* \right)^2 \left(\frac{\partial_x}{\sqrt{e(i\partial_x)}} \phi \right)^2 + \text{c.c.} \right]. \quad (7.22)$$

This is simply one of the terms present in (7.21). It is then also clear which is the analogue of the scalar action (7.14) that would lead to ‘non-relativistic’ action (7.20) with such a quartic term:

$$L_{\text{BDS}} = i\phi^* \partial_t \phi - \frac{1}{2} \phi^* (\partial_t^2 - \bar{\lambda} \partial_x^2) \phi - \frac{\tilde{\lambda}}{4} (\phi^{*2} \phi^2 + \text{c.c.}) + O(\phi^6). \quad (7.23)$$

Omitting here the quadratic term with two time derivatives leads to the standard (leading order) LL action. Since this affects only the quadratic terms, this relation is perfectly consistent with the fact that the BDS ansatz is the minimal (and natural) generalization of the $\text{XXX}_{1/2}$ spin chain, affecting only the dispersion relation. The interaction term in (7.23) is invariant under time-dependent $U(1)$ rotations, so we can also put the kinetic term in (7.23) in the standard relativistic form by applying the rotation $\phi = e^{it} \varphi$ that induces instead a mass term (cf (7.11))

$$L_{\text{BDS}} = -\frac{1}{2} \varphi^* (\partial_t^2 - \bar{\lambda} \partial_x^2 - 1) \varphi - \frac{\tilde{\lambda}}{4} (\varphi^{*2} \varphi^2 + \text{c.c.}) + O(\varphi^6). \quad (7.24)$$

This ‘BDS’ action (7.23) has obviously quite different structure from (7.14) that follows from string theory.

7.3. Strings on $AdS_3 \times S^1$, AFS-type S -matrix in the $SL(2)$ sector and the universality of the dressing phase

An important conclusion of the previous subsection is that the tree-level ‘2-magnon’ S -matrix following from the string action on $R \times S^3$ is indeed the same as the low-energy limit of the AFS Bethe ansatz S -matrix (4.22) in the $SU(2)$ sector. This provided a direct relation between the classical string theory and the low momentum limit of the AFS ansatz. An obvious question is whether this relation can be extended to other sectors. The $SL(2)$ sector is of particular interest: a successful comparison would give a non-trivial check of the suggestion [5, 10] that at the classical level in string theory the dressing phase $\sigma^2 = e^{i\theta_{\text{AFS}}}$ is universal to all sectors of the theory.

The construction of the low momentum limit of the BDS-type scattering matrix in the $SL(2)$ sector proceeds as in section 4 and we postpone the details to appendix A.4. Assuming that the dressing phase is universal, equations (A.20) and (4.17) imply that the ‘tree-level’ part of the low-energy AFS-type S -matrix in the $SL(2)$ sector (i.e. the counterpart of (4.22) and (4.23) in the $SU(2)$ sector) is

$$\begin{aligned} (\tilde{S}^{\text{SL}(2)})_{\text{tree}} &= i \left[(p - p') - \frac{p^2 + p'^2}{pe(p') - p'e(p)} \right] + i\tilde{\theta}_{\text{AFS}} \\ &= -\frac{ipp'}{pe(p') - p'e(p)} [1 + e(p)e(p') - \tilde{\lambda}pp']. \end{aligned} \quad (7.25)$$

In the spirit of the previous subsection, this expression should be compared to the world-sheet tree-level scattering matrix of ‘magnons’ (small string fluctuations near the BMN vacuum in the parametrization (2.11)), i.e. the S^1 geodesic) on $AdS_3 \times S^1$. We shall compute the latter below.

It is worth emphasizing that the notion that the dressing factor σ^2 is universal has a meaning only under the assumption that the vacua in the various sectors are chosen to be the same²⁶. This means, in particular, that to extract the relevant vertex it is necessary to use the uniform gauge [31, 58] as in our discussion of the $SU(2)$ sector (i.e. $t = \tau$, $p_\alpha = \text{const}$, or $\tilde{\alpha} = \frac{J}{\sqrt{\lambda}}\sigma = \frac{x}{\sqrt{\tilde{\lambda}}}$, where α now is the coordinate of S^1 from S^5) and to consider small string fluctuations near the S^1 geodesic.

The corresponding gauge-fixed string action on $AdS_3 \times S^1$, i.e. the counterpart of (7.13), was constructed in [56] (see section 2 there). Using slightly different parametrization than in [56] (see [54]) the metric of $AdS_3 \times \tilde{S}^1$ (after 2d duality $\alpha \rightarrow \tilde{\alpha}$) may be written as $ds^2 = -[dt + B(\vec{\ell})]^2 + d\vec{\ell} d\vec{\ell} + d\tilde{\alpha}^2$. Here $\vec{\ell}$ is a pseudo-unit 3-vector

$$\ell^i \ell^j \eta_{ij} = -1, \quad \eta_{ij} = \text{diag}(-1, 1, 1), \quad (7.26)$$

with a parametrization in terms of a single complex scalar convenient for an expansion near the vacuum $\vec{\ell} = (1, 0, 0)$ being

$$\vec{\ell} = (1 + 2|\phi|^2, -i(\phi - \phi^*)\sqrt{1 + |\phi|^2}, (\phi + \phi^*)\sqrt{1 + |\phi|^2}). \quad (7.27)$$

The connection 1-form B (the analogue of C in (2.1) and (7.13) projected on the world sheet has components

$$B_a = -\frac{1}{2} \int d\xi \epsilon_{ijk} \ell^i \partial_\xi \ell^j \partial_a \ell^k, \quad B_a = \frac{\ell^2 \partial_a \ell^3 - \ell^3 \partial_a \ell^2}{2(1 + \ell^1)} = -\frac{1}{2} (\phi^* \partial_a \phi - \phi \partial_a \phi^*). \quad (7.28)$$

Then the gauge-fixed string action on $AdS_3 \times S^1$ [56] becomes

$$\mathcal{S} = - \int dt \int_0^J dx L, \quad L = \sqrt{-h}, \quad (7.29)$$

$$h = \left[-(1 + B_t)^2 + \frac{1}{4} (\partial_t \vec{\ell})^2 \right] \left[1 - \tilde{\lambda} B_x^2 + \frac{\tilde{\lambda}}{4} (\partial_x \vec{\ell})^2 \right] - \tilde{\lambda} \left[B_x (1 + B_t) - \frac{1}{4} \partial_t \vec{\ell} \cdot \partial_x \vec{\ell} \right]^2.$$

This action has a form of the Nambu action in a static gauge, but, in contrast to (7.13), without a WZ-type term (the analogue of the latter, i.e. B_t , here comes out of the square root term upon ‘fast-string’ expansion leading to the LL action [56]).

With these preliminaries, we are ready to extract the four-point vertex following the same steps as in section 7.1. Expanding (7.29) to quartic order in the fluctuation field ϕ gives

$$L = i\phi^* \partial_t \phi - \frac{1}{2} \phi^* (\partial_t^2 - \tilde{\lambda} \partial_x^2) \phi - \frac{1}{4} [\phi^{*2} (\dot{\phi}^2 - \tilde{\lambda} \phi'^2) + \text{c.c.}] + \frac{1}{4} [i\phi^* \dot{\phi}^* (\dot{\phi}^2 - \tilde{\lambda} \phi'^2) + \text{c.c.}] + \frac{1}{8} (\dot{\phi}^{*2} - \tilde{\lambda} \phi'^{*2}) (\dot{\phi}^{*2} - \tilde{\lambda} \phi'^{*2}) + O(\phi^6). \quad (7.30)$$

The difference compared to the $R_t \times S^3$ case (7.14) is in the change of sign of the second-derivative quartic term (that has to do with the opposite sign of the curvature of AdS_3 compared to S^3) and also in the presence of the 3-derivative term. From here it follows immediately that the quartic vertex for ‘magnons’ with first-order dispersion relation, i.e. the analogue of (7.16) and (7.17), is

$$\mathcal{V}_4^{\text{SL}(2)}(p, p'; k, k') = i \frac{\tilde{\mathcal{V}}_4^{\text{SL}(2)}}{\sqrt{e(p)e(p')e(k)e(k')}}}, \quad (7.31)$$

²⁶ In multi-component integrable field theories, a change of vacuum state typically entails a change of the phase of the scattering matrix. Consequently, changing the vacuum state of only one sector induces a phase change of the scattering matrix of only that sector and thus a relative phase compared to the other sectors.

$$\begin{aligned}
\tilde{\mathcal{V}}_4^{\text{SL}(2)} &= -\tilde{\lambda}(pp' + kk') + [e(p) - 1][e(p') - 1] + [e(k) - 1][e(k') - 1] \\
&\quad + \frac{1}{2}(\tilde{\lambda}pp' - [e(p) - 1][e(p') - 1])(\tilde{\lambda}kk' - [e(k) - 1][e(k') - 1]) \\
&\quad - \frac{1}{2}([e(p) - 1] + [e(p') - 1])(\tilde{\lambda}kk' - [e(k) - 1][e(k') - 1]) \\
&\quad + (p, p') \leftrightarrow (k, k').
\end{aligned} \tag{7.32}$$

The vertex $\mathcal{V}^{\text{SL}(2)}$ simplifies considerably upon multiplication by the delta-function (3.5) enforcing the energy–momentum conservation. It follows then that the tree-level S -matrix of this $SL(2)$ sigma model, i.e. the analogue of (7.18) and (7.19) in the $SU(2)$ case, is

$$\begin{aligned}
\mathcal{V}^{\text{SL}(2)} \frac{\tilde{\lambda}^{-1}e(p)e(p')}{pe(p') - p'e(p)} \delta_+(p, p', k, k') &= (S_{\text{string}}^{\text{SL}(2)})_{\text{tree}} \delta_+(p, p', k, k'), \\
(S_{\text{string}}^{\text{SL}(2)})_{\text{tree}} &= ipp' \frac{\tilde{\lambda}pp' - e(p)e(p') - 1}{pe(p') - p'e(p)}.
\end{aligned} \tag{7.33}$$

Remarkably, this is indeed the same as the ‘tree-level’ part (7.25) of the low-energy AFS-type S -matrix in the $SL(2)$ sector²⁷.

We have therefore shown, in a low energy and strong coupling (classical string) approximation that the dressing phase relating the gauge-theory Bethe ansatz and the world-sheet S -matrix is the *same* in the $SU(2)$ and the $SL(2)$ sectors. This provides a non-trivial test of the proposed [4, 5, 10] generalization of the $SU(2)$ AFS ansatz to other sectors.

7.4. Fermionic $SU(1|1)$ truncation of the superstring action and AFS-type S -matrix

As a further test of the universality of the dressing phase we will now discuss the tree-level S -matrix [8] of the $SU(1|1)$ truncation [53] of the $AdS_5 \times S^5$ superstring action and its comparison with the corresponding phase shift in the AFS-type Bethe ansatz in [4, 5]. The matching of the two S -matrices in the low-energy limit was already noted in [8], but since our perspective is somewhat different and also to clarify some conceptual issues that seem to have more general importance we shall discuss this case in detail below.

Using the low-energy limit of the BDS-type S -matrix in the $SU(1|1)$ sector from appendix A.3 (A.13) and assuming the universality of the AFS phase we find as in (7.25)

$$\begin{aligned}
(\tilde{S}^{\text{SU}(1|1)})_{\text{tree}} &= \frac{i}{2}(p - p') \left[1 - \frac{p - p'}{pe(p') - p'e(p)} \right] + i\tilde{\theta}_{\text{AFS}} \\
&= \frac{i}{2}(p[e(p') - 1] - p'[e(p) - 1]).
\end{aligned} \tag{7.34}$$

This is the expression we would, by analogy with the bosonic sector cases, expect to get as a tree-level S -matrix in the corresponding sector of the superstring theory.

Similarly to the truncation of the string theory to the $SU(2)$ and $SL(2)$ sectors, the $AdS_5 \times S^5$ world-sheet sigma model may be consistently truncated [46, 53] to a fermionic model [53] containing the AdS_5 time coordinate t , two fermionic components Ψ_1, Ψ_2 and a

²⁷ As in the $SU(2)$ sector, this conclusion about the relation of the low-energy AFS S -matrix and classical string sigma model S -matrix is of course consistent with the ‘derivation’ of the $SL(2)$ dressing phase [4] from the classical string model on $AdS_3 \times S^1$ by discretizing the integral equation [8] describing classical solutions of $AdS_3 \times S^1$ string sigma model.

boson α parametrizing an S^1 direction in S^5 . One may then fix the uniform gauge condition [20, 53], i.e. $t = \tau$, $p_\alpha = \frac{J}{\sqrt{\lambda}}$.²⁸

Note that in contrast to the previous two cases of the bosonic sectors where J in the uniform gauge corresponded to the *length* of the spin chain on the gauge-theory side (J was total spin $J_1 + J_2$ in the $SU(2)$ case in (7.13) and the $U(1)$ R -charge in the $SL(2)$ case in (7.29) here J is the bosonic R -charge while the length of the chain is $L = J + \frac{1}{2}M$ where M is the number of fermionic impurities [44, 59] (the corresponding operators are $\text{Tr}(Z^{L-M}\psi^M)$). This suggests a subtlety in the identification of the spin chain and the world-sheet S -matrices in this case (which was indeed already mentioned in [4, 8]).²⁹ Indeed, the S -matrix on the spin-chain side has a meaning only in the context of a specific choice of a ground state and its quantum numbers so the identification of the length on the lhs of the Bethe equations (1.1) with the length of the world-sheet spatial direction is important³⁰.

Using the consistent truncation of [53], T-dualizing the coordinate $\alpha \rightarrow \tilde{\alpha}$ and fixing the gauge $t = \tau$ and $\tilde{\alpha} = \frac{J}{\sqrt{\lambda}}\sigma = \frac{x}{\sqrt{\lambda}}$ one finds the fermionic analogue of the bosonic actions in (7.13) and (7.29), i.e. the action of [53] with each spatial derivative ∂_x having an additional factor of $\sqrt{\lambda}$

$$L = -i\bar{\Psi}(\rho^0\partial_t + \sqrt{\lambda}\rho^1\partial_x)\Psi + \bar{\Psi}\Psi - \frac{1}{4}\sqrt{\lambda}\epsilon^{ab}(\bar{\Psi}\partial_a\Psi\bar{\Psi}\rho^3\partial_b\Psi - \partial_a\bar{\Psi}\Psi\partial_b\bar{\Psi}\rho^3\Psi) + \frac{1}{8}\sqrt{\lambda}\epsilon^{ab}(\bar{\Psi}\Psi)^2\partial_a\bar{\Psi}\rho^3\partial_b\Psi. \quad (7.35)$$

Here Ψ is a two-component spinor (formed from 2 components of the original fermions of the $AdS_5 \times S^5$ superstring), $\bar{\Psi} = \Psi^\dagger\rho^0$ and

$$\rho^0 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \rho^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \rho^3 = \rho^0\rho^1. \quad (7.36)$$

Like the bosonic actions (7.13) and (7.29), this nonlinear fermionic action is classically integrable [53] but should not be expected to be meaningful at the quantum level (in particular, it is not renormalizable; cf [8]): to compute quantum corrections one is to include couplings to all other superstring modes. As in the previous bosonic cases, here we will be interested only in the tree-level 2-particle S -matrix corresponding to (7.35) (where the six-point interaction term may thus be dropped out).

As was pointed out in [46], to compare the truncation of the superstring action to the spin-chain side, the $SU(1|1)$ spin-chain fermionic magnon should be identified with one of the two components of the fermion field $\Psi = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}$. Our aim is thus to compute the tree S -matrix for this component.

Written explicitly in terms of Ψ_1 and Ψ_2 , the Lagrangian (7.35) is, up to the relevant fourth order in the fields,

$$L = -\Psi_1^*(i\partial_t + 1)\Psi_1 - \Psi_2^*(i\partial_t - 1)\Psi_2 + \sqrt{\lambda}\left[\Psi_2^*\partial_x\Psi_1 - \frac{1}{2}\Psi_1^*\Psi_2^*(i\partial_t\Psi_1\partial_x\Psi_1 + i\partial_t\Psi_2\partial_x\Psi_2) + \text{h.c.}\right] + O(\Psi^6) \quad (7.37)$$

²⁸ In the more general sector of string theory, the analogous gauge may be fixed by picking the isometric direction α on S^5 corresponding to the large R -charge, rescaling all other coordinates by $e^{iq_j\alpha}$ such that they become neutral under the $U(1)$ transformation, dualizing $\alpha \rightarrow \tilde{\alpha}$ and setting $t = \tau$, $\tilde{\alpha} = \frac{J}{\sqrt{\lambda}}\sigma$.

²⁹ Let us note also that the dependence of the form of the ‘string’ Bethe ansatz on a choice of world-sheet gauge was emphasized in [12, 60].

³⁰ In general, the gauge choice is also related to the issue of identification of the vacua; the choices of vacua in the three rank-1 sectors are formally the same—the vacuum is the BMN one; however, its embedding in the full string theory may appear to be different.

Since Ψ_1 and Ψ_2 have opposite signs of mass terms, applying the redefinition like that in (7.11) or that relating (7.24) and (7.23), i.e.

$$\Psi_1 = e^{it} \zeta, \quad \Psi_2 = e^{it} \chi, \quad (7.38)$$

we make ζ ‘massless’ while χ more massive, and thus can naturally integrate out the latter. This rotation of the fluctuation fields is necessary in order to relate the truncated string action to a ‘non-relativistic’ LL-type action for the fermionic ‘magnons’ [46] (cf (6.2)). Similar rotation was automatically incorporated in the choice of gauge fixing and field parametrization in the $SU(2)$ (7.13) and $SL(2)$ (7.29) sectors where (7.14) and (7.30) contained linear in time-derivative ‘friction’ term. For comparison, such a term would be absent if one would start with the BMN-type action for bosonic fields³¹.

Solving the equation for χ gives

$$\chi = -\frac{\sqrt{\tilde{\lambda}} \partial_x}{2 - i\partial_t} \zeta + O(\zeta^2) \quad (7.39)$$

and thus the effective Lagrangian for ζ becomes³²

$$\begin{aligned} L(\zeta) = & -\zeta^* \left(i\partial_t - \frac{\tilde{\lambda} \partial_x^2}{2 - i\partial_t} \right) \zeta \\ & - \frac{1}{2} \tilde{\lambda} \left[\zeta^* \frac{\partial_x}{2 + i\partial_t} \zeta^* \left((1 - i\partial_t) \zeta \partial_x \zeta + \tilde{\lambda} \frac{(1 - i\partial_t) \partial_x}{2 - i\partial_t} \zeta \frac{\partial_x^2}{2 - i\partial_t} \zeta \right) + \text{h.c.} \right] + \dots \end{aligned} \quad (7.40)$$

The dispersion relation for ζ is thus $i\partial_t \zeta = (1 \pm \sqrt{1 - \tilde{\lambda} \partial_x^2}) \zeta + O(\zeta^3)$. Using that

$$i\partial_t - \frac{\tilde{\lambda} \partial_x^2}{2 - i\partial_t} = \left(i\partial_t + \sqrt{1 - \tilde{\lambda} \partial_x^2} - 1 \right) P^2, \quad P^2 \equiv \frac{1 + \sqrt{1 - \tilde{\lambda} \partial_x^2} - i\partial_t}{2 - i\partial_t}, \quad (7.41)$$

we can then find the effective Lagrangian for the positive-energy part of ζ with the expected dispersion relation by redefining

$$\zeta = P^{-1} \psi \quad (7.42)$$

getting (cf (6.2) and (7.20))

$$\begin{aligned} \mathcal{L}(\psi) = & -\psi^* (i\partial_t + \sqrt{1 - \tilde{\lambda} \partial_x^2} - 1) \psi - \frac{1}{2} \tilde{\lambda} \left[\bar{P}^{-1} \psi^* \frac{\bar{P}^{-1} \partial_x}{2 + i\partial_t} \psi^* \left((1 - i\partial_t) P^{-1} \psi \partial_x P^{-1} \psi \right. \right. \\ & \left. \left. + \tilde{\lambda} \frac{(1 - i\partial_t) \partial_x P^{-1}}{2 - i\partial_t} \psi \frac{\partial_x^2 P^{-1}}{2 - i\partial_t} \psi \right) + \text{h.c.} \right] + \dots \end{aligned} \quad (7.43)$$

where $\bar{P} = P^\dagger$.

The on-shell 4-vertex corresponding to this Lagrangian (found by eliminating the time derivatives in the quartic term using the free equation of motion for ψ , so that $i\partial_t \rightarrow 1 - e(i\partial_x)$, $P^{-1} \rightarrow \frac{\sqrt{1+e(i\partial_x)}}{\sqrt{2e(i\partial_x)}}$) is (cf (7.16) and (7.17) and (7.31) and (7.32):

$$\begin{aligned} \mathcal{V}^{\text{SU}(1|1)}(p, p'; k, k') &= \frac{i}{2} \frac{\tilde{\mathcal{V}}^{\text{SU}(1|1)}(p, p'; k, k') + \tilde{\mathcal{V}}^{\text{SU}(1|1)}(k, k'; p, p')}{\sqrt{e(p)e(p')e(k)e(k')}}, \\ \tilde{\mathcal{V}}^{\text{SU}(1|1)}(p, p'; k, k') &= -\frac{\tilde{\lambda}}{4} A(k, k') \frac{[p' - p - A(p, p')][(1 + e(k))(1 + e(k')) - \tilde{\lambda} k k']}{\sqrt{(1 + e(p))(1 + e(p'))(1 + e(k))(1 + e(k'))}} \end{aligned} \quad (7.44)$$

³¹ For example, in the case of $R_t \times S^3$ we gauge-fixed the ‘fast’ coordinate which was the combined angle in the two rotational planes (corresponding to the total spin $J_1 + J_2$). Had we fixed, as in the BMN fluctuation case, the angle in only one rotation plane, we would need to apply an extra time-dependent rotation to the fluctuation fields.

³² An equivalent (up to change of notation) quadratic part of the action appeared in the same context in [46].

where $A(k, k') \equiv ke(k') - k'e(k)$. Multiplying this vertex (7.44) by the kinematic factor leads as in (7.18) and (7.33) (with $\delta_+ \rightarrow \delta_-$ as in [8]) to a very simple result for the S -matrix

$$\mathcal{V}^{\text{SU}(1|1)} \frac{\bar{\lambda}^{-1} e(p)e(p')}{pe(p') - p'e(p)} \delta_-(p, p', k, k') = (S_{\text{string}}^{\text{SU}(1|1)})_{\text{tree}}(p', p) \delta_-(p, p', k, k'), \quad (7.45)$$

$$(S_{\text{string}}^{\text{SU}(1|1)})_{\text{tree}} = \frac{i}{2} [pe(p') - p'e(p)].$$

This is the same tree-level S -matrix as one finds from [8] by interpreting their result for the S -matrix of the model of [53] in terms of a ‘non-relativistic’ single-component fermionic field theory.

The string-theory result (7.45) is different from the low momentum limit of the AFS-type S -matrix for the $SU(1|1)$ sector (7.34) by a λ -independent term $\frac{i}{2}(p' - p)$. The difference is actually the necessary correction to the scattering phase appearing from expressing the Bethe equations in terms of the R -charge J rather than the length $L = J + \frac{1}{2}M$ of the chain [4, 8]. On the string-theory side, this shift may be attributed to a choice of the uniform gauge that fixed J instead of L .³³ We conclude that the direct string-theory computations of the ‘magnon’ S -matrix confirm that the non-trivial λ -dependent dressing phase relating the ‘gauge’ and ‘string’ Bethe ansätze is universal for all the rank-1 sectors.

8. Outlook

One important lesson of the present paper is the following. To compare the spin-chain Bethe ansatz phase shift for magnons near the *ferromagnetic* vacuum which have ‘non-relativistic’ first-order dispersion relation to string theory one should re-organize the string-theory S -matrix for BMN-type modes (which originally have relativistic dispersion relation) into the S -matrix for an effective field theory of the positive-energy modes as discussed in section 7.1.

A potential application of the relation between the low-energy limit of the AFS-type S -matrix and the ‘non-relativistic’ form of the classical string action we have investigated in this paper is a possibility of shedding light on the connection between the structure of string $\alpha' \sim \frac{1}{\sqrt{\lambda}}$ corrections and subleading terms in the string phase in (1.2) and (4.11).

There are several open issues that require analysing the complete world-sheet theory at the quantum level. One is relation to quantum corrections within the Landau–Lifshitz framework, e.g., whether (part of) higher order terms in (4.20) may be interpreted as quantum-loop corrections in the LL model (7.20), (7.21). Still, the ‘non-causal’ loops of the LL model do not involve negative-energy modes which are present in the full string loop contributions (where not only quartic but also higher order vertices will be contributing to 2-particle S -matrix), so to account for the latter one needs to go beyond the specific low-energy approximation to the AFS scattering matrix we considered in section 4.

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³³ An alternative interpretation of this change in the scattering phase is that it is due to a difference between the choice of the vacua in the $SU(1|1)$ and the rank-1 bosonic sectors.

Appendix A. Effective field-theory vertices in $SL(2)$ and $SU(1|1)$ sectors

In this appendix, we shall generalize the construction of the effective 2d field-theory vertex from BDS $SU(2)$ -matrix to the other two rank-1 sectors. The same can be repeated also for the AFS-type case (using the general expressions in [5]).

A.1. Bethe ansatz equations

Let us start with recalling the S -matrices that enter the BDS-type Bethe ansätze for $SU(2)$, $SU(1|1)$ and $SL(2)$ sectors [3–5]. The Bethe ansatz equations can be written in the form [5]

$$e^{ip_j L} = \prod_{k \neq j}^M S_\eta(p_j, p_k), \quad S_\eta(p_j, p_k) = \left(\frac{x_j^+ - x_k^-}{x_j^- - x_k^+} \right)^\eta \frac{1 - \frac{\tilde{\lambda}}{4x_j^+ x_k^-}}{1 - \frac{\tilde{\lambda}}{4x_j^- x_k^+}}, \quad (\text{A.1})$$

where $\eta = -1, 0, 1$ for the $SL(2)$, $SU(1|1)$ and $SU(2)$ sectors. Here

$$x_k^\pm = \frac{e^{\pm \frac{i}{2} p_k}}{4 \sin \frac{p_k}{2}} \sqrt{1 + 4\tilde{\lambda} \sin^2 \frac{p_k}{2}}, \quad \tilde{\lambda} \equiv \frac{\lambda}{(2\pi)^2}. \quad (\text{A.2})$$

An important property of the S -matrix in (A.1) is that it can be written as

$$S_\eta(p_j, p_k) = \frac{A_\eta(p_j, p_k) + B_\eta(p_j, p_k)}{A_\eta(p_j, p_k) - B_\eta(p_j, p_k)} = \frac{1 + B_\eta(p_j, p_k)/A_\eta(p_j, p_k)}{1 - B_\eta(p_j, p_k)/A_\eta(p_j, p_k)} \quad (\text{A.3})$$

where B_η is purely imaginary and

$$A_\eta(p_j, p_k) = \frac{1}{2} \left[(x_j^+ - x_k^-)^\eta \left(1 - \frac{\tilde{\lambda}}{4x_j^+ x_k^-} \right) + (x_j^- - x_k^+)^\eta \left(1 - \frac{\tilde{\lambda}}{4x_j^- x_k^+} \right) \right] \quad (\text{A.4})$$

$$B_\eta(p_j, p_k) = \frac{1}{2} \left[(x_j^+ - x_k^-)^\eta \left(1 - \frac{\tilde{\lambda}}{4x_j^+ x_k^-} \right) - (x_j^- - x_k^+)^\eta \left(1 - \frac{\tilde{\lambda}}{4x_j^- x_k^+} \right) \right].$$

A.2. Four-point vertex from S -matrix

Given a field theory of type (2.12), it is a simple exercise to find its tree-level S -matrix. It is related to the four-point vertex $\mathcal{V}(p, p'; k, k')$ in (5.3) by multiplication by the kinematic factor (3.7)

$$S_{\text{tree}} = \mathcal{V}(p, p'; k, k') \frac{\tilde{\lambda}^{-1} e(p) e(p')}{p' e(p) - p e(p')}. \quad (\text{A.5})$$

Here we are free to use the on-shell condition for the momenta p, p', k and k' as well as momentum conservation. Making use of this freedom (which, as discussed above, implies that p, p' equals k, k' or k', k) it is always possible to put the vertex in the form

$$\mathcal{V}(p, p'; k, k') = \frac{1}{2} [\mathcal{V}(p, p') + \mathcal{V}(k, k')]. \quad (\text{A.6})$$

This freedom brings in the issue of reconstructing the off-shell vertex \mathcal{V} from the knowledge of the tree-level S -matrix. This issue is particularly relevant for rank-1 fermion sector. The S -matrix depends on two incoming momenta and as such the vertex will have the structure (A.6). But then it may seem impossible to write a non-trivial Lagrangian L for two anticommuting fields $\psi, \bar{\psi}$ since (A.6) implies that the corresponding terms in L will contain either $\bar{\psi}^2 = 0$ or $\psi^2 = 0$. This is, however, an illusion stemming from a naive use of free equations of motion as well as momentum conservation.

To identify a way to undo the use of momentum conservation we are to take into account the statistics of the scattered particles. In particular, the vertex should be either symmetric or antisymmetric depending whether the scattered particles are bosons or fermions³⁴. Below we will use the momentum conservation constraint on the vertex which is extracted from the scattering matrix in such a way that the required symmetry properties are manifest.

Up to divergent terms that we discard, the 1-loop contribution to the resulting field-theory S -matrix is (cf (5.18))

$$S_{1\text{-loop}} = I_1(p, p') \frac{\tilde{\lambda}^{-1} e(p) e(p')}{p' e(p) - p e(p')}, \quad (\text{A.7})$$

$$I_1(p, p') = (-1)^{|f|} [\mathcal{V}(p, p')]^2 \frac{\tilde{\lambda}^{-1} e(p) e(p')}{p' e(p) - p e(p')}, \quad S_{\text{tree}} = \frac{I_1(p, p')}{\mathcal{V}(p, p')}. \quad (\text{A.8})$$

Then the all-loop S -matrix is

$$S_{\text{all-loop}} = \frac{1 + \frac{1}{2} S_{\text{tree}}}{1 - \frac{1}{2} S_{\text{tree}}}. \quad (\text{A.9})$$

By comparing this with the S -matrix extracted from the Bethe equations (A.3) it follows quite generally that

$$S_{\text{tree}} = 2 \frac{B_\eta(p, p')}{A_\eta(p, p')}, \quad (\text{A.10})$$

$$\begin{aligned} \mathcal{V}(p, p') &= 2\tilde{\lambda} \frac{p' e(p) - p e(p')}{e(p) e(p')} \frac{B_\eta(p, p')}{A_\eta(p, p')} \\ &= 2\tilde{\lambda} \frac{p' e(p) - p e(p')}{e(p) e(p')} \frac{(x_1^+ - x_2^-)^\eta \left(1 - \frac{\tilde{\lambda}}{4x_1^+ x_2^-}\right) - (x_1^- - x_2^+)^\eta \left(1 - \frac{\tilde{\lambda}}{4x_1^- x_2^+}\right)}{(x_1^+ - x_2^-)^\eta \left(1 - \frac{\tilde{\lambda}}{4x_1^+ x_2^-}\right) + (x_1^- - x_2^+)^\eta \left(1 - \frac{\tilde{\lambda}}{4x_1^- x_2^+}\right)} \end{aligned} \quad (\text{A.11})$$

where

$$x_1^\pm = x^\pm(p), \quad x_2^\pm = x^\pm(p').$$

A.3. 4-vertex in the $SU(1|1)$ sector

As explained in the $SU(2)$ case, to compare with the field-theory S -matrix we need to take a specific low-momentum limit in the Bethe ansatz S -matrix, in which $p \rightarrow 0$ and one keeps only the leading term in p at each order in λ . This amounts to the replacement:

$$x^\pm(p) \rightarrow \frac{e(p)}{2p}, \quad e(p) = \sqrt{1 + \tilde{\lambda} p^2}. \quad (\text{A.12})$$

Then

$$\begin{aligned} (\mathcal{S}_{SU(1|1)})_{\text{tree}} &= 2 \frac{\left(1 - \frac{\tilde{\lambda}}{4x_1^+ x_2^-}\right) - \left(1 - \frac{\tilde{\lambda}}{4x_1^- x_2^+}\right)}{\left(1 - \frac{\tilde{\lambda}}{4x_1^+ x_2^-}\right) + \left(1 - \frac{\tilde{\lambda}}{4x_1^- x_2^+}\right)} \\ &\rightarrow (\tilde{\mathcal{S}}_{SU(1|1)})_{\text{tree}} = \frac{i}{2} (p - p') \left[1 - \frac{p - p'}{p e(p') - p' e(p)} \right] \end{aligned}$$

³⁴ This is implicitly taken into account by the fact that the relative sign between the two terms in δ_+ in (3.5) is positive for bosons and negative for fermions (see, e.g., [8]).

$$\begin{aligned}
&= \frac{i\tilde{\lambda}}{4} pp'(p - p') - \frac{i\tilde{\lambda}^2}{16} pp'(p^3 - 2p^2p' + 2pp'^2 - p'^3) \\
&\quad + \frac{i\tilde{\lambda}^3}{32} pp'(p^5 - 2p^4p' + 2p^3p'^2 - 2p^2p'^3 + 2pp'^4 - p'^5) \\
&\quad - \frac{5i\tilde{\lambda}^4}{256} pp'(p^7 - 2p^6p' + 2p^5p'^2 - 2p^4p'^3 \\
&\quad + 2p^3p'^4 - 2p^2p'^5 + 2pp'^6 - p'^7) + \dots
\end{aligned} \tag{A.13}$$

Thus, after dividing by the kinematic factor,

$$\mathcal{V}_{su(1|1)}(p, p') = \frac{i}{2} \tilde{\lambda}(p - p') \left[\frac{pe(p') - p'e(p)}{e(p)e(p')} - \frac{p - p'}{e(p)e(p')} \right] \tag{A.14}$$

i.e.

$$\begin{aligned}
\mathcal{V}_{su(1|1)}(p, p') &= -\frac{i\tilde{\lambda}^2}{4} pp'(p - p')^2 + \frac{i\tilde{\lambda}^3}{16} pp'(p - p')^2(3p^2 + pp' + 3p'^3) \\
&\quad - \frac{i\tilde{\lambda}^4}{32} pp'(p - p')^2(5p^4 + 2p^3p' + 5p^2p'^2 + 2pp'^3 + 5p'^5) \\
&\quad + \frac{i\tilde{\lambda}^5}{256} pp'(p - p')^2(35p^6 + 15p^5p' + 35p^4p'^2 \\
&\quad + 17p^3p'^3 + 35p^2p'^4 + 15pp'^5 + 35p'^6) + \dots
\end{aligned} \tag{A.15}$$

Note that this vertex starts with the 2-loop order λ^2 term, in agreement with the fact that the leading-order LL action in the $SU(1|1)$ sector is free [4, 44–46, 52].

Clearly, the vertex (A.14) does not have the symmetry properties necessary to arise from a fermionic action, as it is symmetric under $p \leftrightarrow p'$. The required antisymmetry is restored by using the momentum conservation to express the overall factor $(p - p')$ in terms of the outgoing momenta:

$$\begin{aligned}
\tilde{\mathcal{V}}_{su(1|1)}(k, k'; p, p') &= \frac{i}{2} \tilde{\lambda}(k - k') \left[\frac{pe(p') - p'e(p)}{e(p)e(p')} - \frac{p - p'}{e(p)e(p')} \right] \\
&= -\frac{i\tilde{\lambda}^2}{4} pp'(p - p')(k - k') + O(\tilde{\lambda}^3).
\end{aligned} \tag{A.16}$$

This vertex allows us to determine the leading part in the quartic interaction term of the resulting fermionic coherent state action corresponding to the BDS-type Bethe ansatz in the $SU(1|1)$ sector (cf (2.12) and (6.2))

$$\mathcal{S} = \int dt \int_0^J dx \{ -\bar{\psi} [i\partial_t + (\sqrt{1 - \tilde{\lambda}\partial_x^2} - 1)] \psi - V(\psi, \bar{\psi}) \}, \tag{A.17}$$

$$V = V_4 + V_6 + \dots, \quad V_4 = \frac{\tilde{\lambda}^2}{4} (\bar{\psi} \psi' \psi' \psi'' + \text{h.c.}) + O(\tilde{\lambda}^3), \tag{A.18}$$

where $\psi' = \partial_x \psi$ and ψ is a complex anticommuting field³⁵. The exact form of V_4 that follows from (A.16) is (cf (5.6))

$$V_4 = \frac{\tilde{\lambda}}{8} \bar{\psi} \partial_x \bar{\psi} \left[\psi \frac{\partial_x}{\sqrt{1 - \tilde{\lambda}\partial_x^2}} \psi - \left(\frac{1}{\sqrt{1 - \tilde{\lambda}\partial_x^2}} \psi \right) \left(\frac{\partial_x}{\sqrt{1 - \tilde{\lambda}\partial_x^2}} \psi \right) \right] + \text{h.c.} \tag{A.19}$$

³⁵ As discussed in [46], to find a similar quadratic term on the string-theory side where one starts with a relativistic massive fermion action one is to solve for one of the two fermionic components in terms of the other.

This interaction term was constructed using the BDS-type S -matrix³⁶. Including a proper ‘string’ phase in the S -matrix one should be able to reconstruct the effective action that should be more closely related to string theory³⁷.

A.4. 4-vertex in the $SL(2)$ sector

Similarly, in the $SL(2)$ sector one finds for the low momentum limit of the BDS-type S -matrix (using (A.10), (A.12), etc)

$$\begin{aligned}
(\mathcal{S}_{sl(2)})_{\text{tree}} &= 2 \frac{(x_1^+ - x_2^-)^{-1} \left(1 - \frac{\tilde{\lambda}}{4x_1^+ x_2^-}\right) - (x_1^- - x_2^+)^{-1} \left(1 - \frac{\tilde{\lambda}}{4x_1^- x_2^+}\right)}{(x_1^+ - x_2^-)^{-1} \left(1 - \frac{\tilde{\lambda}}{4x_1^+ x_2^-}\right) + (x_1^- - x_2^+)^{-1} \left(1 - \frac{\tilde{\lambda}}{4x_1^- x_2^+}\right)} \\
&\rightarrow (\tilde{\mathcal{S}}_{sl(2)})_{\text{tree}} = i \left[(p - p') - \frac{p^2 + p'^2}{pe(p') - p'e(p)} \right] \\
&= \frac{2i pp'}{p - p'} + \frac{i\tilde{\lambda}}{2} \frac{pp'}{p - p'} (p^2 + p'^2) \\
&\quad - \frac{i\tilde{\lambda}^2}{8} \frac{pp'}{p - p'} (p^4 - p^3 p' + 2p^2 p'^2 - 2pp'^3 + p'^4) \\
&\quad + \frac{i\tilde{\lambda}^3}{16} \frac{pp'}{p - p'} (p^6 - p^5 p' + 2p^4 p'^2 - 2p^3 p'^3 + 2p^2 p'^4 - pp'^5 + p'^6) + \dots
\end{aligned} \tag{A.20}$$

Multiplying by the kinematic factor, the exact expression for the vertex summing up the leading small momentum terms at each order in λ expansion is (cf (5.2) and (A.14))

$$\mathcal{V}_{sl(2)}(p, p') = i\tilde{\lambda} \left[(p - p') \frac{pe(p') - p'e(p)}{e(p)e(p')} - \frac{p^2 + p'^2}{e(p)e(p')} \right], \tag{A.21}$$

i.e.

$$\begin{aligned}
\mathcal{V}_{sl(2)}(p, p') &= -2i\tilde{\lambda} pp' + \frac{i\tilde{\lambda}^2}{2} pp' (p^2 + p'^2) - \frac{i\tilde{\lambda}^3}{8} pp' (3p^4 + 5p^3 p' + 5pp'^3 + 3p'^4) \\
&\quad + \frac{i\tilde{\lambda}^4}{16} pp' (5p^6 + 8p^5 p' + 6p^3 p'^3 + 8pp'^5 + 5p'^6) + \dots
\end{aligned} \tag{A.22}$$

The leading-order LL action in the $SL(2)$ sector (depending on a pseudo-unit vector parametrizing AdS_3) was constructed in [54, 55] (see also [57]); the higher order corrections to the spin-chain LL action were not computed before (corrections to the string LL action can be found from the expressions in [56]). The S -matrix approach avoids the problem of complicated explicit form of the dilatation operator in this sector and gives one an efficient method of reconstructing the higher order terms in the effective action. We thus get the following quartic term in the $SL(2)$ effective action (cf (5.6) or (A.19))

$$V_4 = \frac{\bar{\lambda}}{4} \phi^* \phi^* \left[\left(\frac{1}{\sqrt{1 - \bar{\lambda} \partial_x^2}} \phi \right) \frac{\partial_x^2}{\sqrt{1 - \bar{\lambda} \partial_x^2}} \phi + \partial_x \phi \frac{\partial_x}{\sqrt{1 - \bar{\lambda} \partial_x^2}} \phi - \phi \frac{\partial_x^2}{\sqrt{1 - \bar{\lambda} \partial_x^2}} \phi \right] + \text{c.c.} \tag{A.23}$$

As was observed in [4, 5], the BDS S -matrices in the three sectors are formally related by $S_{su(2)} S_{sl(2)} = [S_{su(1|1)}]^2$, implying that

³⁶ It is of course different from the relativistic model of [53] (obtained by a particular truncation of the classical string action of [22]) for which the quantum S -matrix was computed in [8].

³⁷ Let us note again that there is a freedom of off-shell extension that allows us, e.g., to write the denominators here in a more symmetric form, as suggested by the string-theory considerations of section 7 (cf (7.9)).

$$(S_{su(2)})_{\text{tree}} + (S_{sl(2)})_{\text{tree}} = 2(S_{su(1|1)})_{\text{tree}}, \quad (\text{A.24})$$

$$\mathcal{V}_{su(2)} + \mathcal{V}_{sl(2)} = 2\mathcal{V}_{su(1|1)}, \quad (\text{A.25})$$

which provides a simple check on the expressions in (5.2), (A.21) and (A.14). Relation (A.24) is true also after the inclusion of the AFS phase contribution (4.17), which is the same for all sectors.

A.5. Deformation of the BDS S -matrix and the vertex

Let us now comment on the impact of the extra phase that relates the BDS S -matrix S_g and the S -matrix S_s (called, respectively, S_1 and S in (1.2) entering the string Bethe equations on the structure of the resulting field-theory vertex. If we start with

$$S_s(p, p') = S_g(p, p')[\sigma(p, p')]^2, \quad \sigma^2 = e^{i\theta} = \frac{a+b}{a-b}, \quad (\text{A.26})$$

where B in (A.3) and b in (A.26) are assumed to be purely imaginary, then it is not hard to trace this deformation through the steps made above and find the deformed (string)-theory analogues of the quantities A and B in (A.4)

$$\begin{aligned} S_s &= S_g \sigma^2 = \frac{A_g - B_g}{A_g + B_g} \frac{a+b}{a-b} \\ &= \frac{(A_g a - B_g b) - (A_g b - B_g a)}{(A_g a - B_g b) + (A_g b - B_g a)} \equiv \frac{A_s - B_s}{A_s + B_s} \end{aligned} \quad (\text{A.27})$$

We can then reconstruct the four-point vertex (tree S -matrix) for the ‘string’ LL model:

$$V_s \equiv \frac{B_s}{A_s} = \frac{A_g b - B_g a}{A_g a - B_g b} = \frac{v_\theta - V_g}{1 - V_g v_\theta}, \quad V_g = \frac{B_g}{A_g}, \quad v_\theta = \frac{b}{a} = \tan \frac{\theta}{2}, \quad (\text{A.28})$$

where θ is the dressing phase relating the gauge and string theory Bethe ansätze. To relate this to the discussion in section 4 one should apply the small momentum expansion to simplify the entries in (A.27).

Appendix B. Momentum expansion of the leading strong coupling correction to the AFS phase

Here we shall present some details of the small momentum expansion of the leading quantum correction to the AFS phase which we used in section 4 and show that it modifies the small momentum expansion in (4.19) by terms nonanalytic in $\bar{\lambda}$ leading to further deviations from (4.24).

The dressing phase in (1.2), (4.11) is defined by its large $\bar{\lambda}$ expansion

$$\theta = \theta_{\text{AFS}} + \theta_{\text{HL}} + \mathcal{O}\left(\frac{1}{(\sqrt{\bar{\lambda}})^2}\right), \quad (\text{B.1})$$

where θ_{HL} is the first correction in (4.11), (4.12) given by [19]

$$\theta_{\text{HL}} = \sum_{r=2}^{\infty} \sum_{s=r+1}^{\infty} a_{rs} \left(\frac{\bar{\lambda}}{4}\right)^{\frac{r-1}{2} + \frac{s-1}{2}} [q_s(p)q_r(p') - q_s(p')q_r(p)] \quad (\text{B.2})$$

with $a_{rs} = \frac{4}{\pi} \frac{(r-1)(s-1)}{(r-1)^2 - (s-1)^2}$ for odd $r+s$ and 0 otherwise. Using the small momentum limit (4.15) of the charge densities q_r , it is easy to see that θ_{HL} may be written as

$$\theta_{\text{HL}} = pp'g(p)g(p') \sum_{r=2}^{\infty} \sum_{s=r+1}^{\infty} a_{rs} \left[\frac{1}{g(p)^r g(p')^s} - \frac{1}{g(p')^r g(p)^s} \right], \quad (\text{B.3})$$

where

$$g(p) = \frac{\sqrt{\bar{\lambda}} p}{e(p) - 1}$$

is fixed in the small momentum limit. After changing the summation index s to $s = 2n + r + 1$ to take into account the vanishing of a_{rs} for even $r + s$, the double sum in (B.3) equals the second mixed derivative of a double sum computed in [20]:

$$\begin{aligned} \chi_1(x, y) &= \sum_{r=2}^{\infty} \sum_{n=0}^{\infty} \frac{a_{r,2n+r+1}}{(r-1)(2n+r)} \frac{1}{x^{r-1} y^{2n+r}} \\ &= \frac{2}{\pi} \left[\log \frac{y-1}{y+1} \log \frac{x-\frac{1}{y}}{x-y} \right. \\ &\quad \left. + \text{Li}_2 \frac{\sqrt{y}-\frac{1}{\sqrt{y}}}{\sqrt{y}-\sqrt{x}} - \text{Li}_2 \frac{\sqrt{y}+\frac{1}{\sqrt{y}}}{\sqrt{y}-\sqrt{x}} + \text{Li}_2 \frac{\sqrt{y}-\frac{1}{\sqrt{y}}}{\sqrt{y}+\sqrt{x}} - \text{Li}_2 \frac{\sqrt{y}+\frac{1}{\sqrt{y}}}{\sqrt{y}+\sqrt{x}} \right]. \end{aligned} \quad (\text{B.4})$$

Some algebra then leads to

$$\begin{aligned} \theta_{\text{HL}} &= \frac{2pp'g(p)g(p')}{\pi(g(p)-g(p'))(1-g(p)g(p'))} \\ &\quad \times \left[1 - \frac{1}{2} \frac{(1-g(p)g(p'))^2 + (g(p)-g(p'))^2}{(g(p)-g(p'))(1-g(p)g(p'))} \ln \frac{(1+g(p))(1-g(p'))}{(1-g(p))(1+g(p'))} \right] \end{aligned} \quad (\text{B.5})$$

As expected, in the small momentum limit $p \rightarrow 0$ and $\lambda p^2 = \text{fixed}$, the correction θ_{HL} to θ_{AFS} scales as p^2 ; however, its dependence on $\bar{\lambda}$ is nonanalytic:

$$\theta_{\text{HL}} \propto p^2 f(\sqrt{\bar{\lambda}} p). \quad (\text{B.6})$$

Including this additional phase in the (4.19) will modify its low momentum expansion by terms nonanalytic $\bar{\lambda}$ starting with

$$\delta \tilde{S}_{\text{AFS}} = -\frac{i}{3\pi} \bar{\lambda}^{3/2} p^2 p'^2 (p - p') + \dots \quad (\text{B.7})$$

This implies further deviations from the naive expectation (4.24) (expected by analogy with the BDS case) for the low-energy limit of the scattering matrix of the ‘string’ ansatz. It is possible in principle that this nonanalytic dependence on $\bar{\lambda}$ may change once all higher order corrections to θ are resummed. This may be expected on the grounds that the dressing phase should have an analytic expansion at small $\bar{\lambda}$ if it is eventually to agree with perturbative gauge theory.

References

- [1] Minahan J A and Zarembo K 2003 The Bethe-ansatz for $N = 4$ super Yang–Mills *J. High Energy Phys.* JHEP03(2003)013 (Preprint [hep-th/0212208](#))
- [2] Beisert N, Kristjansen C and Staudacher M 2003 The dilatation operator of $N = 4$ super Yang–Mills theory *Nucl. Phys. B* **664** 131 (Preprint [hep-th/0303060](#))
- [3] Beisert N, Dippel V and Staudacher M 2004 A novel long range spin chain and planar $N = 4$ super Yang–Mills *J. High Energy Phys.* JHEP07(2004)075 (Preprint [hep-th/0405001](#))
- [4] Staudacher M 2005 The factorized S -matrix of CFT/AdS *J. High Energy Phys.* JHEP05(2005)054 (Preprint [hep-th/0412188](#))
- [5] Beisert N and Staudacher M 2005 Long-range PSU(2,2|4) Bethe ansatzes for gauge theory and strings *Nucl. Phys. B* **727** 1 (Preprint [hep-th/0504190](#))
- [6] Arutyunov G, Frolov S and Staudacher M 2004 Bethe ansatz for quantum strings *J. High Energy Phys.* JHEP10(2004)016 (Preprint [hep-th/0406256](#))

- [7] Mann N and Polchinski J 2005 Bethe ansatz for a quantum supercoset sigma model *Phys. Rev. D* **72** 086002 (Preprint [hep-th/0508232](#))
- [8] Klose T and Zarembo K 2006 Bethe ansatz in stringy sigma models Preprint [hep-th/0603039](#)
- [9] Gromov N, Kazakov V, Sakai K and Vieira P 2006 Strings as multi-particle states of quantum sigma-models Preprint [hep-th/0603043](#)
- [10] Beisert N 2005 The $su(2|2)$ dynamic S -matrix Preprint [hep-th/0511082](#)
Beisert N 2005 An $SU(1|1)$ -invariant S -matrix with dynamic representations Preprint [hep-th/0511013](#)
- [11] Beisert N and Klose T 2005 Long-range $gl(n)$ integrable spin chains and plane-wave matrix theory Preprint [hep-th/0510124](#)
- [12] Frolov S, Plefka J and Zamaklar M 2006 The $AdS(5) \times S^5$ superstring in light-cone gauge and its Bethe equations Preprint [hep-th/0603008](#)
- [13] Callan C G, Lee H K, McLoughlin T, Schwarz J H, Swanson I J and Wu X 2003 Quantizing string theory in $AdS(5) \times S^5$: beyond the pp-wave *Nucl. Phys. B* **673** 3 (Preprint [hep-th/0307032](#))
Callan C G, McLoughlin T and Swanson I J 2004 Holography beyond the Penrose limit *Nucl. Phys. B* **694** 115 (Preprint [hep-th/0404007](#))
- [14] Serban D and Staudacher M 2004 Planar $N = 4$ gauge theory and the Inozemtsev long range spin chain *J. High Energy Phys.* JHEP06(2004)001 (Preprint [hep-th/0401057](#))
- [15] Beisert N and Tseytlin A A 2005 On quantum corrections to spinning strings and Bethe equations *Phys. Lett. B* **629** 102 (Preprint [hep-th/0509084](#))
- [16] Schafer-Nameki S, Zamaklar M and Zarembo K 2005 Quantum corrections to spinning strings in $AdS(5) \times S^5$ and Bethe ansatz: a comparative study *J. High Energy Phys.* JHEP09(2005)051 (Preprint [hep-th/0507189](#))
- [17] Schafer-Nameki S and Zamaklar M 2005 Stringy sums and corrections to the quantum string Bethe ansatz *J. High Energy Phys.* JHEP10(2005)044 (Preprint [hep-th/0509096](#))
- [18] Janik R A 2006 The $AdS(5) \times S^5$ superstring worldsheet S -matrix and crossing symmetry Preprint [hep-th/0603038](#)
- [19] Hernandez R and Lopez E 2006 Quantum corrections to the string Bethe ansatz Preprint [hep-th/0603204](#)
- [20] Arutyunov G and Frolov S 2006 On $AdS(5) \times S^5$ string S -matrix Preprint [hep-th/0604043](#)
- [21] Freyhult L and Kristjansen C 2006 A universality test of the quantum string Bethe ansatz Preprint [hep-th/0604069](#)
- [22] Metsaev R R and Tseytlin A A 1998 Type IIB superstring action in $AdS(5) \times S(5)$ background *Nucl. Phys. B* **533** 109 (Preprint [hep-th/9805028](#))
- [23] Hofman D M and Maldacena J 2006 Giant magnons Preprint [hep-th/0604135](#)
- [24] Dorey N 2006 Magnon bound states and the AdS/CFT correspondence Preprint [hep-th/0604175](#)
- [25] Berenstein D, Maldacena J M and Nastase H 2002 Strings in flat space and pp waves from $N = 4$ super Yang–Mills *J. High Energy Phys.* JHEP04(2002)013 (Preprint [hep-th/0202021](#))
- [26] Gross D J, Mikhailov A and Roiban R 2002 Operators with large R charge in $N = 4$ Yang–Mills theory *Ann. Phys.* **301** 31 (Preprint [hep-th/0205066](#))
- [27] Santambrogio A and Zanon D 2002 Exact anomalous dimensions of $N = 4$ Yang–Mills operators with large R -charge *Phys. Lett. B* **545** 425 (Preprint [hep-th/0206079](#))
- [28] Berenstein D, Correa D H and Vazquez S E 2006 All loop BMN state energies from matrices *J. High Energy Phys.* JHEP02(2006)048 (Preprint [hep-th/0509015](#))
- [29] Fradkin E H 1991 *Field Theories of Condensed Matter Systems (Frontiers in Physics 82)* (Redwood City, CA: Addison-Wesley) 350 pp
- [30] Kruczenski M 2004 Spin chains and string theory *Phys. Rev. Lett.* **93** 161602 (Preprint [hep-th/0311203](#))
- [31] Kruczenski M, Ryzhov A V and Tseytlin A A 2004 Large spin limit of $AdS(5) \times S^5$ string theory and low energy expansion of ferromagnetic spin chains *Nucl. Phys. B* **692** 3 (Preprint [hep-th/0403120](#))
- [32] Kazakov V A, Marshakov A, Minahan J A and Zarembo K 2004 Classical/quantum integrability in AdS/CFT *J. High Energy Phys.* JHEP05(2004)024 (Preprint [hep-th/0402207](#))
- [33] Ryzhov A V and Tseytlin A A 2004 Towards the exact dilatation operator of $N = 4$ super Yang–Mills theory *Nucl. Phys. B* **698** 132 (Preprint [hep-th/0404215](#))
- [34] Tseytlin A A ‘Semiclassical strings and AdS/CFT *Cargese 2004, String Theory: From Gauge Interactions to Cosmology* pp 265–90 (Preprint [hep-th/0409296](#))
- [35] Minahan J A, Tirziu A and Tseytlin A A 2006 $1/J$ corrections to semiclassical AdS/CFT states from quantum Landau–Lifshitz model *Nucl. Phys. B* **735** 127 (Preprint [hep-th/0509071](#))
- [36] Minahan J A, Tirziu A and Tseytlin A A 2005 $1/J^2$ corrections to BMN energies from the quantum long range Landau–Lifshitz model *J. High Energy Phys.* JHEP11(2005)031 (Preprint [hep-th/0510080](#))
- [37] Tirziu A 2006 Quantum Landau–Lifshitz model at four loops: $1/J$ and $1/J^2$ corrections to BMN energies Preprint [hep-th/0601139](#)

- [38] Rej A, Serban D and Staudacher M 2006 *J. High Energy Phys.* JHEP03(2006)018 (Preprint [hep-th/0512077](#))
- [39] Zarembo K 2006 Antiferromagnetic operators in $N = 4$ supersymmetric Yang–Mills theory *Phys. Lett. B* **634** 552 (Preprint [hep-th/0512079](#))
- [40] Arutyunov G and Tseytlin A A 2006 On highest-energy state in the $SU(1|1)$ sector of $N = 4$ super Yang–Mills theory Preprint [hep-th/0603113](#)
- [41] Beisert N 2005 The dilatation operator of $N = 4$ super Yang–Mills theory and integrability *Phys. Rep.* **405** 1 (Preprint [hep-th/0407277](#))
- [42] Beisert N, Tseytlin A A and Zarembo K 2005 Matching quantum strings to quantum spins: one-loop vs. finite-size corrections *Nucl. Phys. B* **715** 190 (Preprint [hep-th/0502173](#))
- [43] Kruczenski M 2005 Renormalization group flow of reduced string actions Preprint [hep-th/0509178](#)
- [44] Beisert N 2004 The $su(2|3)$ dynamic spin chain *Nucl. Phys. B* **682** 487 (Preprint [hep-th/0310252](#))
- [45] Hernandez R and Lopez E 2004 Spin chain sigma models with fermions *J. High Energy Phys.* JHEP11(2004)079 (Preprint [hep-th/0410022](#))
- [46] Stefanski B and Tseytlin A A 2005 Super spin chain coherent state actions and $AdS(5) \times S^5$ superstring *Nucl. Phys. B* **718** 83 (Preprint [hep-th/0503185](#))
- [47] Mikhailov A 2003 Speeding strings *J. High Energy Phys.* JHEP12(2003)058 (Preprint [hep-th/0311019](#))
Mikhailov A 2005 Slow evolution of nearly-degenerate extremal surfaces *J. Geom. Phys.* **54** 228 (Preprint [hep-th/0402067](#))
- [48] Kruczenski M and Tseytlin A A 2004 Semiclassical relativistic strings in S^5 and long coherent operators in $N = 4$ SYM theory *J. High Energy Phys.* JHEP09(2004)038 (Preprint [hep-th/0406189](#))
- [49] Metsaev R R 2002 Type IIB Green–Schwarz superstring in plane wave Ramond–Ramond background *Nucl. Phys. B* **625** 70 (Preprint [hep-th/0112044](#))
- [50] Frolov S and Tseytlin A A 2002 Semiclassical quantization of rotating superstring in $AdS(5) \times S(5)$ *J. High Energy Phys.* JHEP06(2002)007 (Preprint [hep-th/0204226](#))
- [51] Parnachev A and Ryzhov A V 2002 Strings in the near plane wave background and AdS/CFT *J. High Energy Phys.* JHEP10(2002)066 (Preprint [hep-th/0208010](#))
- [52] Callan C G, Heckman J, McLoughlin T and Swanson I J 2004 Lattice super Yang–Mills: a virial approach to operator dimensions *Nucl. Phys. B* **701** 180 (Preprint [hep-th/0407096](#))
- [53] Alday L F, Arutyunov G and Frolov S 2006 New integrable system of 2dim fermions from strings on $AdS(5) \times S^5$ *J. High Energy Phys.* JHEP01(2006)078 (Preprint [hep-th/0508140](#))
- [54] Stefanski B J and Tseytlin A A 2004 Large spin limits of AdS/CFT and generalized Landau–Lifshitz equations *J. High Energy Phys.* JHEP05(2004)042 (Preprint [hep-th/0404133](#))
- [55] Bellucci S, Casteill P Y, Morales J F and Sochichiu C 2005 $SL(2)$ spin chain and spinning strings on $AdS(5) \times S^5$ *Nucl. Phys. B* **707** 303 (Preprint [hep-th/0409086](#))
- [56] Park I Y, Tirziu A and Tseytlin A A 2005 Spinning strings in $AdS(5) \times S^5$: One-loop correction to energy in $SL(2)$ sector *J. High Energy Phys.* JHEP03(2005)013 (Preprint [hep-th/0501203](#))
- [57] Kazakov V A and Zarembo K 2004 Classical/quantum integrability in non-compact sector of AdS/CFT *J. High Energy Phys.* JHEP10(2004)060 (Preprint [hep-th/0410105](#))
- [58] Arutyunov G and Frolov S 2005 Integrable Hamiltonian for classical strings on $AdS(5) \times S^5$ *J. High Energy Phys.* JHEP02(2005)059 (Preprint [hep-th/0411089](#))
- [59] Callan C G, Heckman J, McLoughlin T and Swanson I J 2004 Lattice super Yang–Mills: a virial approach to operator dimensions *Nucl. Phys. B* **701** 180 (Preprint [hep-th/0407096](#))
- [60] Arutyunov G and Frolov S 2006 Uniform light-cone gauge for strings in $AdS(5) \times S^5$: Solving $su(1|1)$ sector *J. High Energy Phys.* JHEP01(2006)055 (Preprint [hep-th/0510208](#))